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Functional Analysis and its Applications

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Introduction.

Functional analysis is, in short, the study of vector spaces with arbitrary dimension. Developed in the early twentieth century, it is an extension and amalgamation of the fields of complex analysis and linear algebra. David Hilbert, Frigyes and Marcel Riesz, and Stefan Banach were some of the noteworthy mathematicians who pushed the early frontiers of the subject. Many of the ideas produced from their seminal work led to powerful ideas and applications in related subjects: operator theory and ideas of Hilbert spaces were applied to physics in the early part of the twentieth century, and are now of fundamental importance in the field of quantum mechanics, and the theory of partial differential equations would be nearly impossible without the aid of very general fixed-point theorems.

The purpose of this paper is threefold. First, we wish to make clear to the reader the structural similarities and differences between finite- and infinite-dimensional vector spaces. In doing so, the reader with some knowledge of linear algebra and real or complex analysis should be comfortable with the generalizations and abstractions contained below. Second, we wish to explore in detail the mathematical ideas and techniques at the foundation of functional analysis. In other words, we shall develop a context of study and, in doing so, convey to the reader some sense of what one actually means by functional analysis. The proofs are often elegant, and the techniques used to solve problems are quite easily applied to other areas of functional analysis. These two points shall be the topic of the first three chapters. Third, we wish to impress upon the reader the power of functional analysis to solve pure and applied problems in other areas of mathematics. This will be done especially in Chapter 4, where we apply a very general fixed point theorem to a particular partial differential equation to establish existence and smoothness of generalized solutions. This latter work was completed during the author's NSF-funded internship at Cornell University and credit should be also attributed to Phillip Whitman, Frances Hammock, and Alexander Meadows.

Any introductory text on linear algebra concerns itself with vector spaces. In the context of elementary linear algebra, the vector space is finite-dimensional, usually $\mathbb{R}^n$ or $\mathbb{C}^n$. When we say a space X is finite-dimensional, we mean that we can find a finite collection B of linearly independent vectors in X so that any vector in X can be written as a linear combination of vectors in B. We call B a basis for X and the number of elements in B defines the dimension of X.

Linear maps on finite dimensional spaces have nice properties. We say a map $T : X \to Y$ is linear if for all $x, y \in X$ and for all scalars c we have that $T(x + y) = T(x) + T(y)$ and $T(cx) = cT(x)$.[1] A linear map $T : \mathbb{C}^n \to \mathbb{C}^m$ can be represented by an $m \times n$ matrix with complex-valued entries; the fact that the range and domain are finite-dimensional is enough to show that any map which is either one-to-one or onto is automatically a bijective function with a well-defined inverse function which is also a linear map.

In the context of functional analysis, vector spaces are generally infinite-dimensional; that is to say that no finite basis for the spaces exist. For some special infinite-dimensional vector spaces, we will be able to write down countably-infinite generalizations of bases; many interesting and important vector spaces have this property, and the existence of such a countable basis provides enough structure to prove a number of interesting theorems. It is also important to note that linear maps in this context can be one-to-one without being onto, or vice-versa. As an example, consider the vector space $\ell^\infty(\mathbb{C})$, the space of all bounded complex-valued sequences. The backward-shift function $B : \ell^\infty(\mathbb{C}) \to \ell^\infty(\mathbb{C})$ takes a sequence $(a_1, a_2, a_3, ...)$ to the sequence $(a_2, a_3, a_4, ...)$, i.e. it "shifts" the entire sequence to the left, dropping the first term. The fact that B is linear and onto is obvious. However, B maps both $(2, 0, 0, 0, 0, 0, ...)$ and $(1, 0, 0, 0, 0, 0, ...)$ to the sequence of all zeros, proving that B is not one-to-one. The forward shift F takes a sequence $(a_1, a_2, a_3, ...)$ to $(0, a_1, a_2, a_3, ...)$; thus F is one-to-one, but not onto. These two examples indicate that linear maps on infinite-dimensional vector spaces do not behave as nicely as their finite-dimensional relatives.

[1] It may at first seem odd that the second property is not implied by the first. However, consider $T : \mathbb{C} \to \mathbb{C}$ such that $T(z) = \bar{z}$, where $\bar{z}$ denotes the complex conjugate of z. T has the property that $T(z_1 + z_2) = \bar{z}_1 + \bar{z}_2$, but $T(cz) = \bar{c}\bar{z} \neq c\bar{z}$ as long as c has nonzero imaginary part.

To give a meaningful discussion of sequences in and functions on vector spaces of arbitrary dimension, we require some notion of "distance" between two elements of the space. To talk about the convergence of sequences, we will certainly need some way to know when two vectors are close to one another. In the context of the space $\mathbb{R}$, we have a function which tells us the "size" of a real number x: the absolute value function. We then define the distance between two real numbers x and y as $|x - y|$. In the context of functional analysis, this distance will be a function $\|\cdot\|$, called a *norm*, which will map an element of our vector space to a real number.

Definition 1 *Given a vector space X, we shall call a function $\|\cdot\| : X \to \mathbb{R}$ a* norm *if it satisfies the following conditions:*

1. $\|x\| \geq 0$ *for all* $x \in X$
2. $\|x\| = 0$ *if and only if* $x = \mathbf{0}$.
3. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$
4. $\|cx\| = |c| \, \|x\|$ *for all* $x \in X$ *and scalars* c.

Remark. With this definition of distance, we are able to define sequential convergence.

Definition 2 *Given a vector space X and norm $\|\cdot\|$, we say a sequence $\{x_n\}$ in X* converges in norm *(or* converges with respect to $\|\cdot\|$*) to $x \in X$ if, for every positive real number ϵ, there exists N such that $n > N$ implies $\|x_n - x\| < \epsilon$.*

Definition 3 *A sequence $\{x_n\}$ is a* Cauchy sequence *if for every $\epsilon > 0$ there exists N such that $m, n > N$ implies $\|x_n - x_m\| < \epsilon$.*

Remark. In the context of the space $\mathbb{R}$, a sequence is Cauchy if and only if it is convergent (with respect to the norm $|\cdot|$). The fact that convergent sequences are Cauchy is simple to prove: Let $\{x_n\}$ be a sequence converging to x with respect to a norm $\|\cdot\|$. Then for any $\epsilon > 0$, we can choose an N_0 such that $\|x_n - x\| < \frac{\epsilon}{2}$ whenever $n > N_0$. Now, we apply the triangle inequality:

$$\|x_n - x_m\| = \|(x_n - x) + (x - x_m)\| \leq \|x_n - x\| + \|x_m - x\|$$

So, let $m, n > N_0$. Then

$$\|x_n - x_m\| \leq \|x_n - x\| + \|x_m - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which implies $\{x_n\}$ is Cauchy. The converse, that Cauchy sequences are convergent, seems intuitively clear at first, but – as always – we should be cautious of intuition.

Consider, for example, $C^1([-1,1])$, the space of continuously differentiable functions on the interval $[-1,1]$ under the norm $\|f(x)\| = \int_{-1}^{1} |f(x)|\, dx$ (the fact that this is a norm follows easily from the definition). The function $f_n(x) = |x|^{1+\frac{1}{n}}$ is clearly differentiable away from $x = 0$. At $x = 0$, the derivative is well-defined:

$$\lim_{x\to 0^-} \frac{|x|^{1+\frac{1}{n}} - 0}{x - 0} = \lim_{x\to 0^-} -(-x)^{\frac{1}{n}} = 0 = \lim_{x\to 0^+} x^{\frac{1}{n}} = \lim_{x\to 0^+} \frac{|x|^{1+\frac{1}{n}} - 0}{x - 0}$$

Assume $n > m$. Then

$$\begin{aligned}
\left\| |x|^{1+\frac{1}{n}} - |x|^{1+\frac{1}{m}} \right\| &= \int_{-1}^{1} |x|^{1+\frac{1}{n}} - |x|^{1+\frac{1}{m}} = \frac{2n}{2n+1} - \frac{2m}{2m+1} \\
&= \frac{2n(2m+1) - 2m(2n+1)}{(2n+1)(2m+1)} \\
&= 2\frac{n-m}{(2n+1)(2m+1)}
\end{aligned}$$

which we can make arbitrarily small for sufficiently large m, n and so $\{f_n\}$ is Cauchy. If we consider, for the moment, $C^1([-1,1])$ as a subset of $C^0([-1,1])$, the set of continuous functions on $[-1,1]$ under the same norm, we see that $\left\{|x|^{1+\frac{1}{n}}\right\}$ converges in norm to $|x|$:

$$\int_{-1}^{1} \left| |x|^{1+\frac{1}{n}} - |x| \right| dx = 1 - \frac{2n}{2n+1}$$

where the right hand side can be made arbitrarily small for sufficiently large n. The function $|x|$ is certainly not differentiable at $x = 0$ and so $\left\{|x|^{1+\frac{1}{n}}\right\}$ could not possibly be convergent in norm in $C^1([-1,1])$.

The previous example illustrates that Cauchy sequences are not, in general, convergent. The Cauchy property is highly desirable, however, and we often restrict ourselves to normed spaces where all Cauchy sequences are convergent. Spaces where Cauchy sequences are convergent are referred to as being *complete*. A normed linear space which is complete is called a *Banach space*.

It is worth noting here that, since convergence depends on the given norm, we can consider the same vector space X under two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$, where X is complete under $\|\cdot\|_1$ but not complete under $\|\cdot\|_2$. The following chapter will illuminate this subtle distinction, describe a number of important Banach spaces, and also describe an important class of Banach spaces.

Chapter 1.

Consider the set of continuous, real-valued functions on the interval $[0,1]$, denoted $C^0([a,b])$ or just $C([a,b])$. We shall study this space under two different norms. The first, $\|\cdot\|_\infty$, is defined by $\|f(x)\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$ and is commonly referred to as the sup-norm. The second norm will be defined as $\|f(x)\|_2 = \int_a^b |f(x)|^2\, dx$.

Proposition 4 *$C([a,b])$ under $\|\cdot\|_\infty$ is a complete linear space.*

Proof. Using an argument from [4, p. 23], we shall show that $C([a,b])$ is complete with respect to $\|\cdot\|_\infty$. Let $\{f_k\}$ be a Cauchy sequence in $C([a,b])$. Then given an arbitrary positive number ϵ, we have that $\|f_n - f_m\|_\infty < \epsilon$ for sufficiently large m, n. In particular, if we fix any $x \in [a,b]$, then $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \epsilon$. That is, the sequence of real numbers $\{f_n(x)\}$ is Cauchy. Any Cauchy sequence of real numbers is convergent. Thus we have pointwise convergence to a function f where $f(x) = \lim f_n(x)$. Since each f_n is continuous and the uniform limit of continuous functions is again continuous, to show that $\{f_n\}$ converges in norm to f in $C([a,b])$, it suffices to show $f_n \to f$ uniformly. Now, suppose we choose N such that $m, n > N$ and $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. Thus $f_n(x) \in \left(f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}\right)$ for all $x \in [a,b]$. We know that $\lim f_n(x) = f(x)$ is convergent, so it is either in or on the boundary of $\left(f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}\right)$. Thus $f(x) \in \left[f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}\right]$, and so $|f(x) - f_m(x)| \leq \frac{\epsilon}{2} < \epsilon$ and the convergence is uniform, whence $C([a,b])$ is complete with respect to the sup-norm. ■

Proposition 5 *$C([a,b])$ under $\|\cdot\|_2$ is not a complete linear space.*

Proof. We will prove that $C([0,2])$ is not a complete with respect to the given norm; the proof extends in an obvious way to $C([a,b])$. Consider the function

$$f_m(x) = \begin{cases} x^n \text{ for } 0 \leq x \leq 1 \\ 1 \text{ for } 1 \leq x \leq 2 \end{cases}$$

Then $\{f_m\}$ is a decreasing sequence (in the sense that $f_m(x) \geq f_{m+1}(x)$) of positive, square integrable functions. Assume that $n \geq m$. Then

$$\begin{aligned} & \|f_m - f_n\|_2 \\ = & \int_0^1 |x^m - x^n|^2 \, dx + \int_1^2 (1-1)^2 \, dx \\ = & \int_0^1 (x^m - x^n)^2 \, dx \leq \int_0^1 x^{2m} dx \\ = & \frac{1}{2m+1} \end{aligned}$$

which can be made arbitrarily small for large m. Hence $\{f_m\}$ is a Cauchy sequence. If we consider $C([a,b])$ as a subset of the set of bounded functions under the same norm, we see that $f_m \to f$ in norm, where

$$f(x) = \begin{cases} 0 \text{ for } 0 \leq x < 1 \\ 1 \text{ for } 1 \leq x \leq 2 \end{cases}$$

Clearly f is not continuous and so under the $\|\cdot\|_2$ norm, $C([a,b])$ is not complete. ■

Remark. One subtlety has been swept under the carpet in the previous example. When dealing with the integral (whether it be the standard Riemann integral or the Lebesgue integral), the careful reader may note that there is a slight problem in relation to the definition of norm. In the set of bounded functions, the element

$$f(x) = \begin{cases} 1 \text{ for } x = 0 \\ 0 \text{ for } x \neq 0 \end{cases}$$

is certainly bounded and not identically equal to the zero function. Its norm, however, is zero, and clearly violates our requirement that $\|x\| = 0$ if and only if $x = 0$ for any norm. Thus when dealing with the integral, we cannot consider the set of all bounded functions under this norm; instead, we must somehow construct equivalence classes to circumvent this seemingly innocuous – but ultimately crippling – predicament.

Measure theory and Lebesgue integration provide a natural solution to the problem at hand.[2]

[2] The reader interested in more detail should consider a reference on the subject such as [1].

We shall define an equivalence class $[f]$ to be the set of all functions g such that $f - g \equiv 0$ *almost everywhere*; that is to say we will identify two functions which are identically equal except on some very small set (a set of measure zero).

This segue provides an excellent opportunity to present one of the most important spaces of functions: the $L^p(X, d\mu)$ spaces. The choice of the letter L honors the French mathematician Henri Lebesgue. We will define the norm $\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$, called the p-norm, and we will denote with $L^p(X, d\mu)$ the set of all equivalence classes (as discussed above) of integrable functions such that $\int |f|^p \, d\mu < \infty$. We shall here prove that $\|f\|_p$ is a norm and that $L^p(X, d\mu)$ is, in fact, a Banach space when $1 < p < \infty$.

Proposition 6 *For $1 < p < \infty$, $\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$ is a norm on X, a set of integrable functions.*

Proof. That $\|f\|_p \geq 0$ is obvious. That $\|f\| = 0$ if and only if f is identically zero is guaranteed since only the equivalence class including zero could satisfy $\|f\| = 0$. That $\|cf\|_p = |c| \, \|f\|_p$ can be seen since

$$\begin{aligned}
\|cf\|_p &= \left(\int |cf|^p \, d\mu\right)^{1/p} = \left(\int |c|^p \, |f|^p \, d\mu\right)^{1/p} \\
&= \left(|c|^p \int |f|^p \, d\mu\right)^{1/p} = |c| \left(\int |f|^p \, d\mu\right)^{1/p} \\
&= |c| \, \|f\|_p
\end{aligned}$$

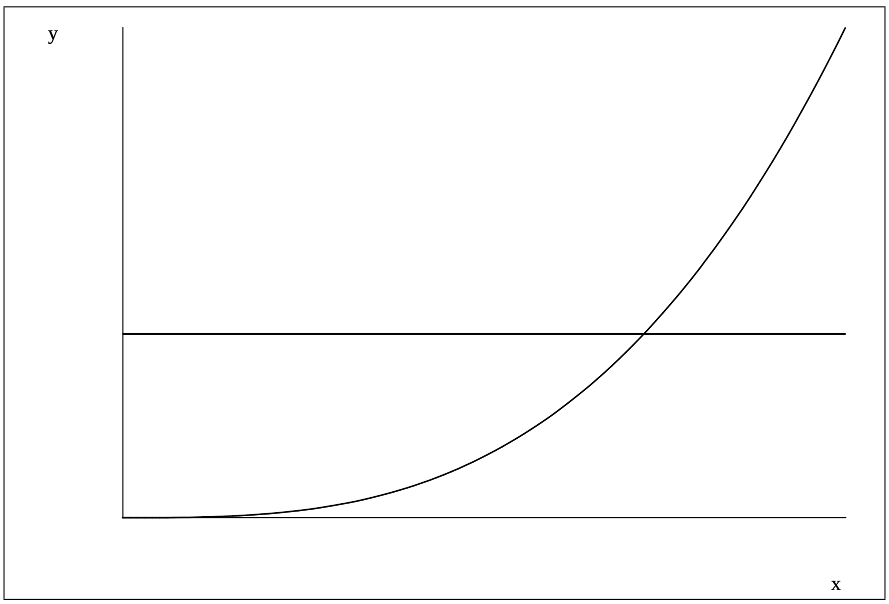
All that remains is to prove that the p-norm satisfies the triangle inequality, which comes from the following theorem. ■

Theorem 7 *(Hölder's Inequality). If p, q are positive reals so that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(X, d\mu)$, $g \in L^q(X, d\mu)$, then $fg \in L^1(X, d\mu)$ and $\|fg\|_1 \leq \|f\|_p \, \|g\|_q$.*

Proof. We shall fill in more completely an argument found in [3, p. 348]. Consider any positive numbers a and b and assume without loss of generality that $a \geq b$. Note that p and $q > 1$. Also note that if $\frac{1}{p}+\frac{1}{q}=1$, then $p+q=pq$ so that

$$(p-1)(q-1)=pq-p-q+1=1$$

If $a^{p-1}<a$, then $\left(a^{p-1}\right)^{q-1}=a<a^{q-1}$. Thus either $a^{p-1}\geq a$ or $a^{q-1}\geq a$. Assume without loss of generality that $a^{p-1}\geq a$, and consider $x=h(y)=y^{q-1}$. Then $y=h^{-1}(x)=x^{p-1}$ since $h\left(h^{-1}(x)\right)=x^{(p-1)(q-1)}=x$. Also, $\int_0^a x^{p-1}dx=\frac{a^p}{p}$ and $\int_0^b y^{q-1}dy=\frac{b^q}{q}$. Since the graph of $y=x^{p-1}$ is increasing and $a^{p-1}\geq a\geq b$, we can assume that the graph of $y=x^{p-1}$ intersects the line $y=b$ on the interval $[0,a]$. The graph then looks as follows:



Graphs of the equation $y=x^{p-1}$ and the line $y=b$ on $[0,a]$

The region in the figure above bounded above by $y=b$ and below by $y=x^{p-1}$ has area $\int_0^b y^{q-1}dy=\frac{b^q}{q}$. The area bounded below $y=x^{p-1}$ is $\int_0^a x^{p-1}dx=\frac{a^p}{p}$. The sum of the two areas, $\frac{b^q}{q}+\frac{a^p}{p}$, is clearly larger than the area of the area below $y=b$, which is ab. Thus $ab\geq\frac{a^p}{p}+\frac{b^q}{q}$ for any positive real numbers a and b. Now, if either f or g is almost everywhere zero, $\|fg\|_1\leq\|f\|_p\|g\|_q$ since both sides of the inequality are zero. Assume now that neither f nor g has norm zero. Let $\tilde{f}=\frac{f}{\|f\|_p}$ and $\tilde{g}=\frac{g}{\|g\|_q}$. So, $\left\|\tilde{f}\right\|_p=\|\tilde{g}\|_q=1$, and by our work above, for any x, we have that

$\left|\tilde{f}(x)\tilde{g}(x)\right| \leq \frac{\left|\tilde{f}(x)\right|^p}{p} + \frac{\tilde{g}(x)^q}{q}$. Integrating both sides, we have

$$\begin{aligned}\int \left|\tilde{f}(x)\tilde{g}(x)\right| &\leq \frac{1}{p}\int \left|\tilde{f}(x)\right|^p + \frac{1}{q}\int \tilde{g}(x)^q \\ &= \frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$

Thus $\int \left|\frac{f}{\|f\|_p}\frac{g}{\|g\|_q}\right| \leq 1$, and so

$$\|fg\|_1 = \int |fg| \leq \|f\|_p \|g\|_q$$

which is Hölder's Inequality. ■

Corollary 8

$$\|f\|_p = \sup\left\{\|fg\|_1 : \|g\|_q = 1\right\}$$

Proof. Using Hölder's Inequality, we now see that if we choose g so that $\|g\|_q = 1$, we have

$$\|f\|_p = \|f\|_p \|g\|_q \geq \|fg\|_1$$

for any such g in $L^q(X, d\mu)$. Thus

$$\|f\|_p \geq \sup\left\{\|fg\|_1 : \|g\|_q = 1\right\}$$

It turns out that we can explicitly find a unit vector g so that equality holds. Consider $g(x) = \frac{f(x)^{p-1}}{\|f\|_p^{p-1}}$. To see that this is, in fact, an element of L^q notice that

$$\left[\int \left(\frac{f(x)^{p-1}}{\|f\|_p^{p-1}}\right)^q d\mu\right]^{1/q} = \frac{1}{\|f\|_p^{p-1}}\left[\int f^{pq-q} d\mu\right]^{1/q}$$

Since $pq - q = p$, we have that

$$\begin{aligned}\left[\int \left(\frac{f(x)^{p-1}}{\|f\|_p^{p-1}}\right)^q d\mu\right]^{1/q} &= \frac{1}{\|f\|_p^{p-1}}\left[\int f^p d\mu\right]^{1/q} \\ &= \frac{\|f\|_p^{p/q}}{\|f\|_p^{p-1}}\end{aligned}$$

which is finite, and so $f(x)^{p-1} / \|f\|_p^{p-1}$ must be in $L^q(X, d\mu)$. Now consider

$$\|fg\|_1 = \int f\frac{f(x)^{p-1}}{\|f\|_p^{p-1}} d\mu = \frac{1}{\|f\|_p^{p-1}}\int f^p d\mu = \frac{\|f\|_p^p}{\|f\|_p^{p-1}} = \|f\|_p$$

and so $\|f\|_p = \sup\left\{\|fg\|_1 : \|g\|_q = 1\right\}$. ■

We are now prepared to prove that the p-norm satisfies the triangle inequality.

Theorem 9 *(Minkowski's Inequality). Let $1 < p < \infty$. Then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.*

Proof. By the above corollary,

$$\begin{aligned} \|f+g\|_p &= \sup\left\{\|(f+g)\,h\|_1 : \|h\|_q = 1\right\} \\ &\leq \sup\left\{\|fh\|_1 + \|gh\|_1 : \|h\|_q = 1\right\} \end{aligned}$$

by the triangle inequality for $\|\cdot\|_1$. And by the properties of sup, we have that

$$\begin{aligned} \|f+g\|_p &\leq \sup\left\{\|fh\|_1 : \|h\|_q = 1\right\} + \sup\left\{\|gh\|_1 : \|h\|_q = 1\right\} \\ &= \|f\|_p + \|g\|_p \end{aligned}$$

And so our proof that $\|\cdot\|_p$ is a valid norm is finished. ■

The only thing holding us back from establishing that $L^p(X, d\mu)$ is, in fact, a Banach space under $\|\cdot\|_p$ for $1 < p < \infty$ is the completeness of $L^p(X, d\mu)$ under $\|\cdot\|_p$. To do so, we shall need to come up with an alternate characterization of completeness.

Theorem 10 *A normed linear space X with norm $\|\cdot\|$ is complete if and only if, for a sequence $\{f_k\}_{k=1}^{\infty}$, the sum $\sum_{k=1}^{\infty} f_k$ converges in X (in norm) whenever $\sum_{k=1}^{\infty} \|f_k\|$ converges.*

Proof. We will borrow from a proof found in [4, p. 62]. To prove the first direction, assume X is complete and that we have a sequence $\{f_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} \|f_k\|$ converges. In other words, for any $\epsilon > 0$ we can choose N such that for $n > N$ we have that $\sum_{k=N}^{\infty} \|f_k\| < \epsilon$. Now, let $n \geq m > N$:

$$\left\|\sum_{k=1}^{n} f_k - \sum_{k=1}^{m} f_k\right\| = \left\|\sum_{k=m+1}^{n} f_k\right\| \leq \sum_{k=m+1}^{n} \|f_k\| \leq \sum_{k=m+1}^{\infty} \|f_k\| < \epsilon$$

Since X is complete and we have shown that $\left\{\sum_{k=1}^{n} f_k\right\}_{n=1}^{\infty}$ is a Cauchy sequence, we know that $\left\{\sum_{k=1}^{n} f_k\right\}_{n=1}^{\infty}$ must converge in X. For the other direction, consider a Cauchy sequence $\{f_k\}_{k=1}^{\infty}$ in

X. Certainly, for each k, we can find a number k_i such that

$$\|f_m - f_n\| < \frac{1}{2^k}, \text{ for } m, n > k_i$$

Then as long as we require $k_{i+1} > k_i$, the sequence $\{f_{k_i}\}_{i=1}^{\infty}$ is a subsequence of $\{f_k\}_{k=1}^{\infty}$. Now, define a sequence as follows; set $g_1 = f_{k_1}$ and $g_i = f_{k_i} - f_{k_{i-1}}$ for $i \geq 2$. Now, the following computation holds:

$$\begin{aligned}
\sum_{i=1}^{l} \|g_i\| &= \|g_1\| + \sum_{i=2}^{l} \left\|f_{k_i} - f_{k_{i-1}}\right\| < g_1 + \sum_{i=2}^{l} \frac{1}{2^{i-1}} \\
&< g_1 + \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = g_1 + 1
\end{aligned}$$

Hence $\left\{\sum_{i=1}^{l} \|g_i\|\right\}$ is a bounded, increasing sequence of real numbers, so it must converge, and so we know that $\left\{\sum_{i=1}^{l} g_i\right\}$ must converge in norm by our hypothesis. Since $\sum_{i=1}^{n} g_i = f_{k_n}$, let f denote the limit of the subsequence $\{f_{k_i}\}_{i=1}^{\infty}$. We know that $\{f_k\}_{k=1}^{\infty}$ is Cauchy, so for any $\epsilon > 0$, we can find an N such that

$$\|f_m - f_n\| < \frac{\epsilon}{2} \text{ whenever } m, n > N$$

and also so that

$$\|f_{k_i} - f\| < \frac{\epsilon}{2} \text{ whenever } k_i > N$$

Choose such an N, and note that

$$\|f_k - f\| \leq \|f_k - f_{k_i}\| + \|f - f_{k_i}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Whence $\{f_k\}_{k=1}^{\infty}$ is convergent, and our proof is done. ■

This previous proof provides us with the tools to prove the following theorem.

Theorem 11 *(Riesz–Fisher). $L^p(X, d\mu)$ under $\|\cdot\|_p$ is a Banach space.*

Proof. By our previous work, we need only show that $L^p(X, d\mu)$ is complete under $\|\cdot\|_p$. The details of the proof depend heavily on the previous theorem, but involve a number of important theorems from Lebesgue measure theory (Fatou's Lemma and Lebesgue's Dominated Convergence Theorem, in particular), of which the author has carefully avoided discussion. The interested reader should refer to [4, p. 63] for a complete proof. ■

Remark. To the case $p = 1$, that is for the space $L^1(X, d\mu)$ under the norm $\|f\|_1 = \int |f| \, d\mu$, the above theorems can be extended using the above techniques and the triangle inequality for the $|\cdot|$ function. We denote by $L^\infty(X, d\mu)$ the set of bounded integrable functions under the sup-norm $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. To this space, the above theorems also extend.

Remark. The $L^p(X, d\mu)$ spaces for $1 \le p \le \infty$ are interesting spaces to study. Spaces of integrable functions provide sufficient varieties of objects to study without being so general that one cannot prove interesting theorems. It is also worth noting that many differential equations, in which there is enormous practical interest, can be related to integral equations. These integral equations can sometimes be represented as linear operators on integrable functions and the $L^p(X, d\mu)$, or subsets thereof, are often the best spaces in which to look for solutions. It is also the case that when we consider our space to be $X = \mathbb{N}$ (although X could be any discrete space) and the counting measure on X (that is if $A \subset X$, then $m(A) = |A|$ if A is finite and ∞ otherwise), then $\int_{\mathbb{N}} f = \sum_{n=0}^{\infty} |f(n)|$. We denote by ℓ^p the set of sequences $\{f(n)\}$ which are p-summable (i.e. $\sum_{n=0}^{\infty} |f(n)|^p < \infty$). There is significant difficulty in finding the actual sum of of a series, and so, by using the Lebesgue integral, any properties we can obtain from general L^p spaces can often be used to study infinite series.

Remark. It is important to note here that the character of the L^p spaces depends greatly upon the flavor of the space X and the measure. The following four examples will illustrate this point.

Example 1. Let $p < q$. Our goal is to give an example of a measure space $(X, d\mu)$ such that $L^p(X, \mu) \subset L^q(X, \mu)$, where the containment is proper. Take the measure space to consist of the set of positive integers $\mathbb{N}$, along with the counting measure. The integral of the function f, in this case, is then simply the sum of f ranging over $\mathbb{N}$, i.e. $\int f d\mu = \sum\limits_{n=1}^{\infty} |f(n)|$. The spaces $L^p(\mathbb{N}, \mu)$

and $L^q(\mathbb{N},\mu)$ consist of all functions such that $\sum\limits_{n=1}^{\infty}|f(n)|^p$ and $\sum\limits_{n=1}^{\infty}|f(n)|^q$, respectively, are finite. Since $p<q$, it follows that $\frac{q}{p}>1$. Thus if we let $f\in L^p(X,\mu)$, we see that $\sum\limits_{n=1}^{\infty}|f(n)|^p$ is finite.

Lemma 12 *Let $f\in L^p(\mathbb{N},\mu)$. Then the sequence $\{|f(n)|^p\}_1^{\infty}$ contains only finitely many values greater than or equal to one.*

Proof. Assume that this claim does not hold. Then $|f(n)|^p\geq 1$ for infinitely many values of n, whence the sum $\sum\limits_{n=1}^{\infty}|f(n)|^p$ could not possibly be finite. ■

Claim 13 *If $\sum\limits_{n=1}^{\infty}|f(n)|^p$ is finite and $\frac{q}{p}>1$, then $\sum\limits_{n=1}^{\infty}|f(n)|^q$ is also finite.*

Proof. By Lemma 12, there is some N such that $|f(n)|^p<1$ for $n>N$. Then consider the sequence $\{|f(n)|^p\}_N^{\infty}$. Since $\frac{p}{q}>1$, it follows that $|f(n)|^p>(|f(n)|^p)^{q/p}=|f(n)|^q$. Thus $\sum\limits_{n=N}^{\infty}|f(n)|^q<\sum\limits_{n=N}^{\infty}|f(n)|^p$ and, since $\sum\limits_{n=N}^{\infty}|f(n)|^p$ converges and is finite, $\sum\limits_{n=N}^{\infty}|f(n)|^q$ is also convergent and finite by the comparison test. Thus $\sum\limits_{n=1}^{\infty}|f(n)|^q=\sum\limits_{n=1}^{N-1}|f(n)|^q+\sum\limits_{n=N}^{\infty}|f(n)|^q$ is finite. Hence $L^p(\mathbb{N},\mu)\subset L^q(\mathbb{N},\mu)$. To show the containment is proper, consider $\frac{1}{n^{1/p}}$. We know by the power test that $\sum\frac{1}{n^{q/p}}$ is finite since $q/p>1$. Thus $\frac{1}{n^{1/p}}\in L^q(X,\mu)$. However, $\sum\frac{1}{n^{p/p}}=\sum\frac{1}{n}$ is infinite and hence $\frac{1}{n^{1/p}}\notin L^p(\mathbb{N},\mu)$. ■

Example 2. We wish to find a measure space such that $L^q\subset L^p$ if $p<q$, where the containment is proper. Consider the unit interval with Lebesgue measure. In this space, the integral is simply the Riemann integral. Now, let $f\in L^q$. Let E be the subset of $[0,1]$ such that $f(x)>1$ for every element x in E. Since $p<q$, it follows that $f^p<f^q$ for values of x in E and hence $\int_E|f|^p\,d\mu\leq\int_E|f|^q\,d\mu$. We know the function $g(x)=1$ has finite integral over $[0,1]$ and hence has finite integral over $[0,1]-E$. Since $f^p\leq 1$ on $E-[0,1]$, we see that $\int\limits_{[0,1]-E}|f|^p\,d\mu\leq\int\limits_{[0,1]-E}1d\mu<\infty$. Thus

$$\int\limits_{[0,1]}|f|^p\,d\mu=\int_E|f|^p\,d\mu+\int\limits_{[0,1]-E}|f|^p\,d\mu\leq\int_E|f|^q\,d\mu+\int\limits_{[0,1]-E}1d\mu<\infty$$

and so $f \in L^p$, whence $L^q \subset L^p$. To show the containment is proper, consider the function $f(x) = \frac{1}{x^{1/q}}$. Since $\frac{p}{q} < 1$, we see that $\int\limits_0^1 \left(\frac{1}{x^{1/q}}\right)^p dx = \int\limits_0^1 \frac{1}{x^{p/q}} dx < \infty$ by the power test. However, $\int\limits_0^1 \left(\frac{1}{x^{1/q}}\right)^q dx = \int\limits_0^1 \frac{1}{x} dx$ is not finite.

Example 3. We now demonstrate a measure space such that $L^p \not\subset L^q$ and $L^q \not\subset L^p$ when $p < q$. Consider $\mathbb{R} - \mathbb{R}^-$ with the Borel measure. As stated above, the integral for this measure space is identical to the Riemann integral. Consider the functions $f(x) = \left\{ \begin{array}{c} 0 \text{ if } x < 1 \\ \frac{1}{x^{1/p}} \text{ if } x \geq 1 \end{array} \right\}$. The integral is then

$$\int\limits_0^\infty |f(x)|^q \, dx = \int\limits_1^\infty \frac{1}{x^{q/p}} dx = \left(\frac{1}{\left(1 - \frac{q}{p}\right) x^{\frac{q}{p}-1}} \right)_1^\infty = \frac{1}{\frac{q}{p} - 1} < \infty$$

since $\frac{q}{p} > 1$. However, $\int\limits_0^\infty |f(x)|^p \, dx = \int\limits_1^\infty \frac{1}{x} dx$ which is not finite. Thus $L^q \not\subset L^p$. Conversely, consider $f(x) = \left\{ \begin{array}{c} \frac{1}{x^{1/q}} \text{ if } x < 1 \\ 0 \text{ if } x \geq 1 \end{array} \right\}$. Then

$$\int\limits_0^\infty |f(x)|^p \, dx = \int\limits_0^1 \frac{1}{x^{p/q}} dx = \frac{1}{1 - p/q} < \infty$$

but $\int\limits_0^\infty |f(x)|^p \, dx = \int\limits_0^1 \frac{1}{x} dx$ is not finite. Thus $L^p \not\subset L^q$.

Example 4. Lastly, we will give an example where $L^p = L^q$ for all p and q. Consider the set $\{1, 2, 3\}$ with the counting measure. The integral is now the sum over $\{1, 2, 3\}$. Since Let f be any function from $\{1, 2, 3\}$ to $\mathbb{R}$ which takes on finite values. Then $f(1)^x, f(2)^x$, and, $f(3)^x$ are finite for all $x > 0$. Thus $f(1)^x + f(2)^x + f(3)^x$ is finite for all $x > 0$ and we have that if $f(1)^p + f(2)^p + f(3)^p$ each element of the sum is finite, each element of the sum to the q/p power is finite and, finally, $f(1)^q + f(2)^q + f(3)^q$ is finite. Thus $L^p \subset L^q$. A similar argument proves that $L^q \subset L^p$, and hence $L^p = L^q$.

Chapter 2.

Having discussed Banach spaces, a highly general class of spaces, we will focus our attention on a more specialized class of spaces, which turn out to have particularly desirable qualities and – in a number of contexts – provide answers to a variety of fairly old problems.

Definition 14 *A vector space X is called an* inner product space *if there is a function* $(\cdot,\cdot) : X \times X \to \mathbb{C}$ *with the following properties for all f, g, h in X and α in $\mathbb{C}$*

1. $(f, f) \geq 0$ *and* $(f, f) = 0$ *if and only if* $f = \mathbf{0}$.
2. $(f + g, h) = (f, h) + (g, h)$
3. $(\alpha f, g) = \alpha (f, g)$
4. $(f, g) = \overline{(g, f)}$

We call such a function an inner product.[3]

Example. Consider $C[a, b]$, the space of complex valued functions on the interval $[a, b]$, and define our inner product to be, for $f, g \in C[a, b]$:

$$(f, g) = \int_a^b f(x)\, \overline{g(x)} dx$$

Properties 2,3, and 4 of the inner product are immediately satisfied. Property 1 is satisfied since $f(x)\, \overline{f(x)} = |f(x)|^2 \geq 0$, and so

$$(f, f) = \int_a^b f(x)\, \overline{f(x)} dx = \int_a^b |f(x)|^2\, dx \geq \int_a^b 0\ dx = 0$$

To see that

$$\int_a^b |f(x)|^2\, dx = 0 \text{ if and only if } f(x) \equiv 0$$

[3] It is important to note that, while there is general agreement in the mathematical community that $(\alpha f, g) = \alpha (f, g)$, physicists would say $(\alpha f, g) = \bar{\alpha} (f, g)$ and instead that $(f, \alpha g) = \alpha (f, g)$. In quantum mechanics, a long-standing notation is $(\psi_\alpha, \psi_\beta) = \int_a^b \psi_\alpha^* \psi_\beta dx$, so the conjugated function is the first function, not the second. This can lead only to minor confusion, but it is an important distinction to be aware of.

consider $A_\alpha = \left\{x \in [a,b] : |f(x)|^2 > \alpha\right\}$. If $f \neq 0$ then, for some α_0, the set A_{α_0} must not be measure zero, or else $f = 0$ almost everywhere and $f = 0$. Since the integral is additive,

$$\begin{aligned}
\int_a^b |f(x)|^2 \, dx &= \int_{A_{\alpha_0}} |f(x)|^2 \, dx + \int_{[a,b]-A_{\alpha_0}} |f(x)|^2 \, dx \\
&\geq \int_{A_{\alpha_0}} |f(x)|^2 \, dx + 0 \\
&> \alpha_0^2 m(A_{\alpha_0}) > 0
\end{aligned}$$

So, if $\int_a^b |f(x)|^2 \, dx = 0$ then $f = 0$ (almost everywhere). If $f = 0$, then $\int_a^b |f(x)|^2 \, dx$ is obviously zero and so $\int_a^b f(x) \overline{g(x)} dx$ is a well-defined inner product. One should notice something about this inner product:

$$(f, f) = \int f(x) \overline{f(x)} dx = \int |f(x)|^2 \, dx = \|f\|_2^2$$

Thus $L^2(X, d\mu)$ is an inner product space. Anyone who has studied the standard dot product in $\mathbb{R}^n$ should not be too surprised that $\sqrt{(f,f)}$ tells us the "size" of f.

Theorem 15 $\|f\| = \sqrt{(f,f)}$ *is a norm.*

Proof. By definition, $(f,f) \geq 0$, so $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$. For any α, $\|\alpha f\|^2 = (\alpha f, \alpha f) = \alpha(f, \alpha f) = \alpha\bar{\alpha}(f,f) = |\alpha|^2 (f,f)$ and so $\|\alpha f\| = |\alpha| \|f\|$. To show that $\|f + g\| \leq \|f\| + \|g\|$, consider

$$\begin{aligned}
\|f+g\|^2 &= (f+g, f+g) = (f, f+g) + (g, f+g) \\
&= (f,f) + (f,g) + (g,f) + (g,g) \\
&= (f,f) + (g,g) + (f,g) + (f,g)^* \\
&= (f,f) + (g,g) + 2\operatorname{Re}(f,g) \\
&\leq (f,f) + (g,g) + 2|(f,g)|
\end{aligned}$$

So long as $|(f,g)| \leq \|f\| \, \|g\| = \sqrt{(f,f)}\sqrt{(g,g)}$, a result we shall prove below, we have that

$$\begin{aligned}
\|f+g\|^2 &\leq (f,f)+(g,g)+2\,|(f,g)| \\
&\leq (f,f)+(g,g)+2\sqrt{(f,f)}\sqrt{(g,g)} \\
&= \left(\sqrt{(f,f)}+\sqrt{(g,g)}\right)^2
\end{aligned}$$

and so

$$\|f+g\| \leq \|f\|+\|g\|$$

Thus $\sqrt{(f,f)}$ is indeed a norm. ■

Definition 16 *We call two vectors f, g* orthogonal *if $(f,g)=0$. We call a set of vectors $\{x_n\}$ an* orthonormal set *if $(x_m, x_n) = \delta_{mn}$, where δ denotes the Kronecker delta function.*

Theorem 17 *(Pythagorean Theorem). Let $\{f_n\}_{n=1}^N$ be an orthonormal set in our vector space X. For all f in X, the following equality holds:*

$$(f,f) = \sum_{n=1}^{N} |(f,f_n)|^2 + \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n, f - \sum_{n=1}^{N}(f_n,f)\,f_n\right)$$

Proof. (from [3, p. 37]). First, we write $f = \sum_{n=1}^{N}(f,f_n)\,f_n + \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right)$. Now,

$$\begin{aligned}
&\left(f - \sum_{n=1}^{N}(f,f_n)\,f_n, \sum_{n=1}^{N}(f,f_n)\,f_n\right) \\
&= \sum_{n=1}^{N}(f,f_n)\,(f,f_n) - \left(\sum_{n=1}^{N}(f,f_n)\,f_n, \sum_{n=1}^{N}(f,f_n)\,f_n\right)
\end{aligned}$$

In the second term, all cross terms go to zero since $(f_m, f_n) = 0$ when $m \neq n$. Thus

$$\begin{aligned}
&\left(f - \sum_{n=1}^{N}(f,f_n)\,f_n, \sum_{n=1}^{N}(f,f_n)\,f_n\right) \\
&= \sum_{n=1}^{N}(f,f_n)\,(f,f_n) - \left(\sum_{n=1}^{N}(f,f_n)\,f_n, \sum_{n=1}^{N}(f,f_n)\,f_n\right) \\
&= \sum_{n=1}^{N}(f,f_n)\,(f,f_n) - \sum_{n=1}^{N}(f,f_n)^2\,(f_n,f_n) \\
&= \sum_{n=1}^{N}(f,f_n)\,(f,f_n) - \sum_{n=1}^{N}(f,f_n)^2 \\
&= 0
\end{aligned}$$

since $(f_n, f_n) = 1$. Now,

$$\begin{aligned}
(f,f) &= \left(\sum_{n=1}^{N}(f,f_n)\,f_n + \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right), \sum_{n=1}^{N}(f,f_n)\,f_n + \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right)\right) \\
&= \left(\left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right), \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right)\right) \\
&\quad + \left(\left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right), \sum_{n=1}^{N}(f,f_n)\,f_n\right) \\
&\quad + \left(\sum_{n=1}^{N}(f,f_n)\,f_n, \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right)\right) \\
&\quad + \left(\sum_{n=1}^{N}(f,f_n)\,f_n, \sum_{n=1}^{N}(f,f_n)\,f_n\right)
\end{aligned}$$

The second and third inner products are zero by our computation above. Also,

$$\begin{aligned}
&\left(\sum_{n=1}^{N}(f,f_n)\,f_n, \sum_{n=1}^{N}(f,f_n)\,f_n\right) = \sum_{m=1}^{N}\left[(f,f_m)\left(f_m, \sum_{n=1}^{N}(f,f_n)\,f_n\right)\right] \\
&= \sum_{m=1}^{N}\left[(f,f_m)\left(f_m,(f,f_m)\,f_m\right)\right] = \sum_{m=1}^{N}(f,f_m)\,\overline{(f,f_m)}\,(f_m,f_m) \\
&= \sum_{m=1}^{N}(f,f_m)\,\overline{(f,f_m)} = \sum_{m=1}^{N}|(f,f_m)|^2
\end{aligned}$$

Consequently,

$$\begin{aligned}
(f,f) &= \left(\sum_{n=1}^{N}(f,f_n)\,f_n, \sum_{n=1}^{N}(f,f_n)\,f_n\right) + \left(\left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right), \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right)\right) \\
&= \sum_{n=1}^{N}|(f,f_n)|^2 + \left(\left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right), \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right)\right)
\end{aligned}$$

which completes the proof. ■

Corollary 18 *(Bessel's Inequality). Let $\{f_n\}_{n=1}^{N}$ be an orthonormal set in our vector space X. Then*

$$(f,f) \geq \sum_{n=1}^{N}|(f,f_n)|^2$$

Proof. Since $(x,x) \geq 0$, $\left(\left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right), \left(f - \sum_{n=1}^{N}(f,f_n)\,f_n\right)\right) \geq 0$ and the above inequality follows immediately. ■

Corollary 19 *(Schwarz's Inequality). Let* $\{f_n\}_{n=1}^{N}$ *be as defined in Corollary 18. Then*

$$|(f,g)| \leq \sqrt{(f,f)}\sqrt{(g,g)}$$

Proof. The case where $f=0$ is trivial, so assume $f \neq 0$. The set $\left\{ \frac{f}{\sqrt{(f,f)}} \right\}$ is, then, an orthonormal set (albeit trivially). So,

$$\begin{aligned} (g,g) &\geq \left| \left(\frac{f}{\sqrt{(f,f)}}, g \right) \right|^2 \\ &= \frac{1}{(f,f)} \left|(f,g)\right|^2 \end{aligned}$$

and so $\left|(f,g)\right|^2 \leq (g,g)\,(f,f)$. ■

Remark. What we have just shown is that any inner product space is automatically a normed vector space with the norm

$$\|\cdot\| = \sqrt{(\cdot,\cdot)}$$

We call $\|\cdot\|$ the *norm induced by* $(\cdot,\cdot)$.

Definition 20 *We will call any complete inner product space which is complete under the induced norm a* Hilbert space*.*

Remark. The space $l^2(\mathbb{R})$ is a Hilbert space under the inner product

$$(\{x_n\}, \{y_n\}) = \sum_{n=0}^{\infty} x_n y_n$$

We already know that $l^2(\mathbb{R})$ is complete, so we need only verify that it is an inner product space under the above inner product. Note that

$$(\{x_n\}, \{x_n\}) = \sum_{n=0}^{\infty} x_n^2 \geq 0$$

and that we have equality only in the case where $x_n = 0$ for all n.

Clearly, $(\alpha x_n, y_n) = \alpha (x_n, y_n)$, and property 4 is satisfied trivially. The final property is true, since

$$\begin{aligned} (\{x_n + y_n\}, \{z_n\}) &= \sum_{n=0}^{\infty} (x_n + y_z) z_n \\ &= \sum_{n=0}^{\infty} x_n z_n + \sum_{n=0}^{\infty} y_n z_n \\ &= (\{x_n\}, \{z_n\}) + (\{y_n\}, \{z_n\}) \end{aligned}$$

where we are allowed to split up the addition since eventually $x_n z_n$ is convergent by $(x_n + z_n)^2 = x_n^2 + 2x_n z_n + z_n^2$, which converges since $\{x_n + z_n\}$, $\{x_n\}$, and $\{z_n\}$ are all in ℓ^2 and are thus square-summable. So, $x_n z_n$ has a convergent upper bound, namely $\left\{(x_n + z_n)^2 - x_n^2 - y_n^2\right\}$ and must be convergent. Hence $\ell^2(\mathbb{R})$ is a Hilbert space. We could, in fact, replace the space $\mathbb{R}$ with $\mathbb{C}$ to make $l^2(\mathbb{C})$, the set of complex-valued square-summable sequences, and replace the inner product with $(\{x_n\}, \{y_n\}) = \sum_{n=0}^{\infty} x_n \overline{y_n}$; under this inner product, $l^2(\mathbb{C})$ is a Hilbert space.

By similar reasoning, we can also show that $L^2(X, d\mu)$ is a Hilbert space. It turns out, though, that this is the only L^p space which is a Hilbert space. To see why this is true, we prove the following theorem.

Theorem 21 *(Parallelogram Law). On a vector space V, a norm $\|\cdot\|$ is induced by an inner product $(\cdot,\cdot)$ if and only if for all u, v in V,*

$$2\|u\|^2 + 2\|v\|^2 = \|u + v\|^2 + \|u - v\|^2$$

Proof. Assume our norm is induced by an inner product. Then

$$2\|u\|^2 + 2\|v\|^2 = 2(u, u) + 2(v, v)$$

and

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= (u+v, u+v) + (u-v, u-v) \\
&= (u,u) + (u,v) + (v,u) + (v,v) \\
&\quad + (u,u) + (u,-v) + (-v,u) + (-v,-v) \\
&= 2\,(u,u) + 2\,(v,v) + (u, v-v) + (v-v, u) \\
&= 2\,(u,u) + 2\,(v,v)
\end{aligned}$$

Thus

$$2\,\|u\|^2 + 2\,\|v\|^2 = \|u+v\|^2 + \|u-v\|^2$$

Now assume that the parallelogram equality holds for all x, y. We define our inner product by

$$(x,y) = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 - i\left(\|x+iy\|^2 - \|x-iy\|^2\right)\right]$$

Now,

$$\begin{aligned}
(x,x) &= \frac{1}{4}\left[\|x+x\|^2 - \|x-x\|^2 - i\,\|x+ix\|^2 + i\,\|x-ix\|^2\right] \\
&= \frac{1}{4}\left[4\,\|x\|^2 - i\,|1+i|\,\|x\|^2 + i\,|(1-i)|\,\|x\|^2\right] \\
&= \frac{1}{4}\left[4\,\|x\|^2 + i\sqrt{2}\,\|x\|^2 - i\sqrt{2}\,\|x\|^2\right] \\
&= \|x\|^2 \geq 0
\end{aligned}$$

with equality if and only if $x = 0$ by the properties of the norm. To see that $(y,x) = \overline{(x,y)}$, notice that

$$\begin{aligned}
(x,y) &= \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 - i\left(\|x+iy\|^2 - \|x-iy\|^2\right)\right] \\
(y,x) &= \frac{1}{4}\left[\|y+x\|^2 - \|y-x\|^2 - i\left(\|x+iy\|^2 - \|x-iy\|^2\right)\right] \\
&= \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 - i\,\|i\,(-iy+x)\|^2 + i\,\|i\,(-iy-x)\|^2\right] \\
&= \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 + i\,\|x-iy\|^2 - i\,\|-iy-x\|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2 - i\left(\|x-iy\|^2 - \|x+iy\|^2\right)\right] \\
&= \overline{(x,y)}
\end{aligned}
$$

To show that

$$
\begin{aligned}
(\alpha x, y) &= \frac{1}{4}\left[\|\alpha x+y\|^2 - \|\alpha x-y\|^2 - i\left(\|\alpha x+iy\|^2 - \|\alpha x-iy\|^2\right)\right] \\
&= \alpha\,(x,y)
\end{aligned}
$$

is actually fairly difficult. First, one must prove that the relation holds for integers by using induction. To see that the relation holds for $\alpha = 2$, note that

$$
\begin{aligned}
\|2x+y\|^2 &= \|x+(x+y)\|^2 \\
&= 2\|x\|^2 + 2\|(x+y)\|^2 - \|y\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|2x-y\|^2 &= \|x+(x-y)\|^2 \\
&= 2\|x\|^2 + 2\|(x-y)\|^2 - \|y\|^2
\end{aligned}
$$

So,

$$
\begin{aligned}
4\,\mathrm{Re}\,(2x,y) &= \|2x+y\|^2 - \|2x-y\|^2 \\
&= 2\|x\|^2 + 2\|(x+y)\|^2 - \|y\|^2 - \left(2\|x\|^2 + 2\|(x-y)\|^2 - \|y\|^2\right) \\
&= 2\|(x+y)\|^2 - 2\|(x-y)\|^2 = 4\,\mathrm{Re}\,[2\,(x,y)]
\end{aligned}
$$

and likewise,

$$
-i\|2x+iy\|^2 + i\|2x-iy\|^2 = 2i\|x+iy\|^2 - 2i\|x-iy\|^2
$$

so that

$$
(2x,y) = 2\,(x,y)
$$

The induction step, then, is

$$
\begin{aligned}
((n+1)\,x, y) &= (nx, y) + (x, y) \\
&= n\,(x, y) + (x, y) \\
&= (n+1)\,(x, y)
\end{aligned}
$$

where $((n+1)\,x, y) = (nx, y) + (x, y)$ will be proved later. So the relation indeed holds for any integer. To see that it holds for any rational, notice that

$$n\left(\frac{1}{n}x, y\right) = \left(\left(n \cdot \frac{1}{n}\right) x, y\right) = (x, y)$$

So, dividing by n yields

$$\left(\frac{1}{n}x, y\right) = \frac{1}{n}\,(x, y)$$

Hence

$$\left(\frac{m}{n}x, y\right) = m\left(\frac{1}{n}x, y\right) = \frac{m}{n}\,(x, y)$$

and we have proven the relation holds for the the rationals as well. Now, we merely use the density of the rationals in the reals[4]. Let α be any real number and let α_n be a sequence of rationals which converges to α. Then

$$\alpha\,(x, y) = (\lim \alpha_n)\,(x, y) = \lim\left[\alpha_n\,(x, y)\right] = \lim\,(\alpha_n x, y) = (\alpha x, y)$$

Now, proving that the relation holds for all complex numbers simply relies on this result, the linearity of the inner product, and the fact that a complex number z can be written as $a + bi$ where a and b are real. To see that $(x + z, y) = (x, y) + (z, y)$, note the following by applying the parallelogram

[4] The proof that the reals are dense in the rationals follows directly from either from the Archimedean Principle (which implies that between any two real numbers there is a rational number) or the existence of decimal expansions for every real number.

law

$$\begin{aligned}
4\,\mathrm{Re}\,(x+z,y) &= \|x+z+y\|^2 - \|x+z-y\|^2 \\
4\,\mathrm{Re}\,[(x,y)+(z,y)] &= \|x+y\|^2 + \|z+y\|^2 - \left(\|x-y\|^2 + \|z-y\|^2\right) \\
&= \frac{1}{2}\left[\|x+y+z+y\|^2 + \|x+y-(z+y)\|^2\right] \\
&\quad -\frac{1}{2}\left[\|x-y+z-y\|^2 + \|x-y-(z-y)\|^2\right] \\
&= \frac{1}{2}\left[\|x+2y+z\|^2 + \|x-z\|^2\right] \\
&\quad -\frac{1}{2}\left[\|x+z-2y\|^2 + \|x-z\|^2\right] \\
&= \frac{1}{2}\left[\|x+2y+z\|^2 - \|x+z-2y\|^2\right]
\end{aligned}$$

Using a manipulation similar to the one we used in showing $(nx,y) = n\,(x,y)$, we have that

$$\begin{aligned}
4\,\mathrm{Re}\,[(x,y)+(z,y)] &= \frac{1}{2}\left[\|x+2y+z\|^2 - \|x+z-2y\|^2\right] \\
&= \frac{1}{2}\left[2\,\|x+z+y\|^2 - 2\,\|x+z-y\|^2\right] \\
&= \|x+z+y\|^2 - \|x+z-y\|^2 \\
&= 4\,\mathrm{Re}\,(x+z,y)
\end{aligned}$$

A similar result holds for the imaginary part of $(x+z,y)$ and so we have that

$$(x+z,y) = (x,y)+(z,y)$$

and we have proven that the above formula is indeed an inner product whenever the parallelogram law applies. ■

We are now prepared to show that the only L^p space which is a Hilbert space is L^2.

Proposition 22 *Of the L^p spaces, only L^2 is a Hilbert space (except when the measure space is trivial).*

Proof. We shall prove this for $L^p\left([0,2]\,,dx\right)$; the proof of the general result is almost the same. We know that the Parallelogram Law must hold for any norm which is induced by an inner product.

In particular, it must hold for

$$u = \begin{cases} A \text{ if } x \in [0,1] \\ B \text{ if } x \in [1,2] \end{cases}$$

and

$$v = \begin{cases} A \text{ if } x \in [0,1] \\ B \text{ if } x \in [1,2] \end{cases}$$

where A and B are positive real numbers. So,

$$\begin{aligned} 2\|u\|^2 + 2\|v\|^2 &= \|u+v\|^2 + \|u-v\|^2 \\ 2\left(\int_0^2 |u|^p\,dx\right)^{2/p} + 2\left(\int_0^2 |v|^p\,dx\right)^{2/p} &= \left(\int_0^2 |u+v|^p\,dx\right)^{2/p} + \left(\int_0^2 |u-v|^p\,dx\right)^{2/p} \end{aligned}$$

The left-hand side becomes

$$\begin{aligned} & 2\left(\int_0^2 |u|^p\,dx\right)^{2/p} + 2\left(\int_0^2 |v|^p\,dx\right)^{2/p} \\ = \;& 2\left(\int_0^1 |u|^p\,dx + \int_1^2 |u|^p\,dx\right)^{2/p} + 2\left(\int_0^1 |v|^p\,dx + \int_1^2 |v|^p\,dx\right)^{2/p} \\ = \;& 2\left(A^p + B^p\right)^{2/p} + 2\left(A^p + B^p\right)^{2/p} = 4\left(A^p + B^p\right)^{2/p} \end{aligned}$$

and the right-hand side is

$$\begin{aligned} & \left(\int_0^2 |u+v|^p\,dx\right)^{2/p} + \left(\int_0^2 |u-v|^p\,dx\right)^{2/p} \\ = \;& \left(\int_0^2 |A+B|^p\,dx\right)^{2/p} + \left(\int_0^2 |A-B|^p\,dx\right)^{2/p} \\ = \;& \left(2\,|A+B|^p\right)^{2/p} + \left(2\,|A-B|^p\right)^{2/p} \\ = \;& 2^{2/p}\,|A+B|^2 + 2^{2/p}\,|A-B|^2 \\ = \;& 2\cdot 2^{2/p}\left(A^2 + B^2\right) \end{aligned}$$

So, for equality to take place,

$$4\left(A^p + B^p\right)^{2/p} = 2\cdot 2^{2/p}\left(A^2 + B^2\right)$$

or, if $A^p + B^p \neq 0$

$$\begin{aligned} 2\left(A^p + B^p\right)^{2/p} &= 2^{2/p}\left(A^2 + B^2\right) \\ \frac{A^p + B^p}{A^2 + B^2} &= 2^{\frac{2}{p}-1} \end{aligned}$$

and so

$$\ln\left(\frac{A^p + B^p}{A^2 + B^2}\right) = \left(\frac{2}{p} - 1\right)\ln 2$$

If A and B are greater than 1 and $p > 2$, then $\frac{A^p+B^p}{A^2+B^2} > 1$. The logarithm of a number greater than one is always positive, so the left-hand side is positive in this case. On the right-hand side, though, $\left(\frac{2}{p} - 1\right)\ln 2$ is negative whenever $p > 2$. So, $p > 2$ is impossible. If A and B are greater than 1 and $p < 2$, then $0 < \frac{A^p+B^p}{A^2+B^2} < 1$ and the logarithm of a number less than one is always negative so the left-hand side is negative. The right-hand side, however, is $\left(\frac{2}{p} - 1\right)\ln 2 > 0$. Thus $p < 2$ is also impossible. The only possible case left is $p = 2$. This yields $0 = 0$, as we expect, since the inner product

$$(f, g) = \int_0^2 f(x)\overline{g(x)}dx$$

indeed induces $\|\cdot\|_2$. ■

Remark. The above work shows that many norms are not easy to induce by an inner product. The L^p spaces are fairly nice, but the fact that L^p is a Hilbert space for only $p = 2$ often makes L^2 the most attractive of the L^p spaces in which to work.

We move away from the previous discussion to generalize two results from finite-dimensional vector spaces. In finite-dimensional vector spaces, we can always decompose any vector into a sum of two vectors which are perpendicular to one another. This tool is often used to compute distance formulas in $\mathbb{R}^3$. To compute the distance between two parallel planes, one takes the vector projection of any vector between the two planes onto a unit normal to the planes. This property hides the subtle fact that any vector between the two planes has two components: one along the plane and one perpendicular to it. The second property of finite-dimensional vector spaces which we should like to generalize is that of a basis. In $\mathbb{R}^n$, a basis is a set of n linearly independent vectors which span $\mathbb{R}^n$. In infinite-dimensional spaces, we require infinitely many vectors to span the space and so the notion of basis must be generalized somewhat. The fact that we can extend the notion of basis to Hilbert spaces is one of the most powerful tools in functional analysis.

Consider a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{M}$ of $\mathcal{H}$. We can define another subspace $\mathcal{M}^\perp$ to be the set of all vectors $x \in \mathcal{H}$ such that $(x, y) = 0$ whenever $y \in \mathcal{M}$. To see that $\mathcal{M}^\perp$ is a subspace, one must simply apply the basic properties of the inner product. Without too much trouble, one can see that $\left(\mathcal{M}^\perp\right)^\perp = \mathcal{M}$. (If we consider $x \in \mathcal{M}^\perp$, then $(x, y) = 0$ for y in $\mathcal{M}$, so $\left(\mathcal{M}^\perp\right)^\perp \subset \mathcal{M}$. If $x \in \mathcal{M}$, then $(x, y) = 0$ whenever $x \in \mathcal{M}^\perp$ and so $\mathcal{M} \subset \left(\mathcal{M}^\perp\right)^\perp$, hence $\left(\mathcal{M}^\perp\right)^\perp = \mathcal{M}$.) It turns out that the spaces $\mathcal{M}$ and $\mathcal{M}^\perp$ define our space $\mathcal{H}$. The following theorems explain why. Proofs can be found in [3, p. 42].

Theorem 23 *If $\mathcal{M}$ is a closed subspace of a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$, then there is a unique $y \in \mathcal{M}$ such that $\|x - y\| \leq \|x - z\|$ for all z in $\mathcal{M}$.*

Theorem 24 *If $\mathcal{M}$ and $\mathcal{H}$ are as in the previous theorem, then any x in $\mathcal{H}$ can be written uniquely as $x = y + z$, where $y \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$.*

Remark. The previous two theorems provide an example of a property shared by inner products on finite-dimensional and infinite-dimensional vector spaces. In $\mathbb{R}^3$, for example, if we take $\mathcal{M}$ to be any line through the origin, then $\mathcal{M}^\perp$ is simply a plane through the origin which is normal to that line; any vector in $\mathbb{R}^3$ can then be written as the vector sum of an element from $\mathcal{M}$ and an element from $\mathcal{M}^\perp$, and this representation is unique. Unlike $\mathbb{R}^3$, we generally cannot find a set of finitely many basis vectors for a Hilbert space. However, the following discussion will generalize the notion of basis, so that we can often find a countably infinite basis for a Hilbert space.

Definition 25 *We call an orthonormal set $\{x_n\}$ in a Hilbert space $\mathcal{H}$ an* orthonormal basis *for $\mathcal{H}$ if is maximal; that is $\{x_n\}$ is not contained in any other orthonormal set.*

Remark. We cannot guarantee that every Hilbert space has an orthonormal basis unless we accept the axiom of choice. It is not difficult to prove the existence of an orthonormal basis using the axiom of choice.

Theorem 26 *(Parseval's Identity). If $\mathcal{H}$ is a Hilbert space and $\{x_\alpha\}_{\alpha\in A}$ is an orthonormal basis for $\mathcal{H}$, then for any x in $\mathcal{H}$,*

$$x = \sum_{\alpha\in A} (x_\alpha, x)\, x_\alpha$$

and

$$\|x\|^2 = \sum_{\alpha\in A} |(x_\alpha, x)|^2$$

Proof. We use an argument from [3, 45]. By Bessel's Inequality, for any finite subset A' of A, we see that

$$\sum_{\alpha\in B} |(x_\alpha, x)|^2 \leq \|x\|^2$$

So, we know $|(x_\alpha, x)| > 0$ for at most a union of finite subsets of A (or else $\sum_{\alpha\in A} |(x_\alpha, x)|^2$ could not possibly be finite). This union, at the very worst, is countable. Denote this set as $B = \{x_k : |(x_k, x)| > 0\}$. Then the sequence $\left\{\sum_k |(x_k, x)|^2\right\}$ is bounded and monotone increasing and thus converges. If we consider the sequence $x_n = \sum_{k=1}^{n} (x_k, x)\, x_k$, we see that, assuming $m > n$ sufficiently large,

$$\begin{aligned}
\|x_n - x_m\|^2 &= \left\| \sum_{k=n+1}^{m} (x_k, x)\, x_k \right\|^2 \\
&= \left(\sum_{k=n+1}^{m} (x_k, x)\, x_k, \sum_{k=n+1}^{m} (x_k, x)\, x_k \right) \\
&= \sum_{k=n+1}^{m} |(x_k, x)|^2 < \epsilon
\end{aligned}$$

where the last equality uses the fact that $\{x_k\}$ is an orthonormal set and the inequality holds since $\left\{\sum_k |(x_k, x)|^2\right\}$ converges for large enough m and n. Then $x_n \to x_0$, by the completeness of our

Hilbert space. Now,

$$\begin{aligned}
(x - x_0, x_m) &= \lim_{n\to\infty} \left(x - \sum_{k=1}^{n} (x_k, x)\, x_k, x_m \right) \\
&= (x - x, x_m) = 0
\end{aligned}$$

For $x_\alpha \neq x_k$ for any k, we have that

$$\begin{aligned}
(x - x_0, x_\alpha) &= \lim_{n\to\infty} \left(x - \sum_{k=1}^{n} (x_k, x)\, x_k, x_\alpha \right) \\
&= (x, x_\alpha) - \lim_{n\to\infty} \sum_{k=1}^{n} (x_k, x)\,(x_k, x_\alpha) \\
&= 0
\end{aligned}$$

Thus $x - x_0$ is orthogonal to all x_α, and so $x - x_0 = 0$. Hence

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} (x_k, x)\, x_k$$

and we have that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} (x_k, x)\, x_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |(x_k, x)|^2 \\
&= \|x\|^2 - \sum_{\alpha\in A} |(x_k, x)|^2
\end{aligned}$$

which completes our proof. ■

Definition 27 *We shall call an orthonormal sequence* $\{x_n\}$ complete *if it is an orthonormal basis.*

This definition differs subtely from the orthonormal basis, since the sequence is countable.

Theorem 28 *(Parseval's Theorem). Let* $\{f_n\}$ *be an orthonormal sequence in* $\mathcal{H}$*, a Hilbert space. Then* $\{f_n\}$ *is complete if and only if for every* f *in* $\mathcal{H}$*, we have that*

$$\|f\|^2 = \sum_{k=1}^{n} |(f_k, f)|^2$$

Proof. By our previous theorem, we have the first direction already. To see the second direction, assume

$$\|f\|^2 = \sum_{k=1}^{n} |(f_k, f)|^2$$

for all f in $\mathcal{H}$. Define $s_n = \sum_{k=1}^{n} (f_k, f)\, f_k$. Then

$$\begin{aligned} \|s_n - f\|^2 &= |(s_n - f, s_n - f)| \\ &= (s_n, s_n) - (f, s_n) - (s_n, f) + (f, f) \\ &= \|s_n\|^2 + \|f\|^2 - \|s_n\|^2 - \|s_n\|^2 \\ &= \|f\|^2 - \|s_n\|^2 \end{aligned}$$

By definition

$$\begin{aligned} \lim_{n\to\infty} \|s_n - f\|^2 &= \lim_{n\to\infty} \|f\|^2 - \|s_n\|^2 \\ &= \|f\|^2 - \|f\|^2 = 0 \end{aligned}$$

and so $\lim s_n = f$, which implies f is complete. ■

Remark. It turns out that the representation $f = \sum_{k=1}^{\infty} (f_k, f)\, f_k$ is unique. That is, if $f = \sum_{k=1}^{\infty} c_k f_k$, then $c_k = (f_k, f)$. The argument to show this is true is similar in nature to the one we just illustrated and can be found in [4, p. 81].

We shall complete this section by elaborating on an a classic example.

Theorem 29 *$C([-\pi, \pi])$ is dense in $L^2([-\pi, \pi], dx)$*

Proof. We shall prove this theorem by considering the following set

$$\left\{ \frac{\cos nx}{\sqrt{\pi}} : n = 1, 2, ... \right\} \cup \left\{ \frac{\sin nx}{\sqrt{\pi}} : n = 1, 2, ... \right\} \cup \left\{ \frac{1}{\sqrt{2\pi}} \right\}$$

This set is clearly a subset of $C([-\pi, \pi])$, and we claim that it is an orthonormal sequence in $L^2([-\pi, \pi], dx)$. Consider

$$\begin{aligned} \left(\frac{\cos nx}{\sqrt{\pi}}, \frac{\cos kx}{\sqrt{\pi}} \right) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \cos kx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} [\cos (m+n) x + \cos (m-n) x]\, dx \end{aligned}$$

by adding the trigonometric identities:

$$\begin{aligned}\cos(a+b) &= \cos a\cos b-\sin a\sin b\\ \cos(a-b) &= \cos a\cos b+\sin a\sin b\end{aligned}$$

Thus,

$$\begin{aligned}\text{if } m &\neq n,\text{ then } \left(\frac{\cos nx}{\sqrt{\pi}},\frac{\cos mx}{\sqrt{\pi}}\right)=0\\ \text{if } m &= n,\text{ then } \left(\frac{\cos nx}{\sqrt{\pi}},\frac{\cos mx}{\sqrt{\pi}}\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}dx=1\end{aligned}$$

Similar application of trigonometric identities yield the same result for $\left(\frac{\sin nx}{\sqrt{\pi}},\frac{\sin mx}{\sqrt{\pi}}\right)$ and $\left(\frac{\sin nx}{\sqrt{\pi}},\frac{\cos mx}{\sqrt{\pi}}\right)$. Also,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\cos nx dx &= 0\\ \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\sin nx dx &= 0\\ \int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}dx &= 1\end{aligned}$$

and so this set is indeed an orthonormal set. To prove the set is complete is significantly trickier; an excellent proof can be found in [4, p. 86]. Thus any element of $L^2([-\pi,\pi],dx)$ can be written as a countable sum of elements of a proper subset of $C([-\pi,\pi])$, and so $C([-\pi,\pi])$ is dense in $L^2([-\pi,\pi],dx)$. ■

Remark. That $\left\{\frac{\cos nx}{\sqrt{\pi}},\frac{\sin nx}{\sqrt{\pi}},\frac{1}{\sqrt{2\pi}}:n=1,2,...\right\}$ is a complete orthonormal basis for L^2 is one of the heralded theorems of applied mathematics. Combined with a number of other orthonormal sets (the Laguerre and Legendre polynomials, for instance), the physicist is able to solve a number of differential equations via series solution. In that method, an infinite linear combination of orthonormal elements is computed with the coefficients unknown. When a function is expanded in terms of $\sin x$ and $\cos x$, the series that results is called a Fourier series, named after the French mathematician who developed the entire program discussed above.

Remark. Fourier series can be used to compute infinite series. Consider $f(x) = x$. Then

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx dx &= 0\\ \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} x dx &= 0\end{aligned}$$

since both x and $x\sin kx$ are odd functions. Integration by parts yields

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx dx &= -2\pi\frac{1}{\pi}\frac{1}{k}\cos k\pi\\ &= 2\frac{1}{k}(-1)^{1+k}\end{aligned}$$

Thus

$$x = 2\sum_{k=1}^{\infty}\frac{(-1)^{1+k}}{k}\sin kx$$

Applying Parseval's Identity, we have

$$\begin{aligned}\|x\|^2 &= \left\|2\sum_{k=1}^{\infty}\frac{(-1)^{1+k}}{k}\sin kx\right\|^2\\ &= 4\sum_{k=1}^{\infty}\left(\frac{(-1)^{1+k}}{k}\sin kx, \frac{(-1)^{1+k}}{k}\sin kx\right)\\ &= 4\sum_{k=1}^{\infty}\frac{1}{k^2}(\sin kx, \sin kx)\\ &= 4\pi\sum_{k=1}^{\infty}\frac{1}{k^2}\end{aligned}$$

Now,

$$\|x\| = \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3}$$

So,

$$\begin{aligned}\frac{2\pi^3}{3} &= 4\pi\sum_{k=1}^{\infty}\frac{1}{k^2}\\ \sum_{k=1}^{\infty}\frac{1}{k^2} &= \frac{\pi^2}{6}\end{aligned}$$

A similar method, using $f(x) = x^2$, shows that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$$

In fact, $\sum_{k=1}^{\infty}\frac{1}{k^{2n}}$ can be computed exactly for any natural number n.

Chapter 3.

We have spent the past two sections defining and studying a number of different classes of spaces. In this section, we shall define a general class of functions and discuss the properties of mappings between Banach spaces.

Let X and Y be normed linear spaces. Recall that a function $T : X \to Y$ is a *linear transformation* from X to Y if, for all x, y in X and scalars c,

$$\begin{aligned} T(x+y) &= T(x) + T(y) \\ T(cx) &= cT(x) \end{aligned}$$

In the special case where $Y = \mathbb{C}$, the function T is called a *linear functional.*

Definition 30 *Let X and Y be normed linear spaces and let $T : X \to Y$ be a function. Then we define*

$$\|T\| = \sup_{x \in X - \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}$$

where $\|\cdot\|_Y$ denotes the norm on Y and $\|\cdot\|_X$ the norm on X. If T is a linear map and $x \neq 0$, then

$$\begin{aligned} \|T\| &= \sup_{x \in X - \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X} = \sup_{x \in X - \{0\}} \left\| T\left(\frac{x}{\|x\|_X} \right) \right\|_Y \\ &= \sup_{\|x\|_X = 1} \|T(x)\|_Y \end{aligned}$$

We say T is bounded if $\|T\| < \infty$. We say T is norm-attaining *if there is a unit vector x such that $\|T(x)\| = \|T\|$.*

We have already seen the backward and forward shifts and have observed that they are linear transformations. We shall compute their norms and produce a pair of other important examples.

Example. Recall that the backward shift $B : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ is defined by

$$B(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots)$$

and so

$$\begin{aligned}\|B(a_1,a_2,a_3,...)\| &= \|(a_2,a_3,a_4,...)\| \\ &= \sum_{k=2}^{\infty}|a_k|^2 \\ &\leq \sum_{k=1}^{\infty}|a_k|^2 \\ &= \|(a_1,a_2,a_3,...)\|\end{aligned}$$

Thus it holds that

$$\frac{\|B(a_1,a_2,a_3,...)\|}{\|(a_1,a_2,a_3,...)\|} \leq 1$$

i.e. $\|B\| \leq 1$. If we select the sequence $\left(0,2^{-1/2},2^{-1},2^{-3/2},2^{-2},...\right)$, then

$$\left\|\left(0,2^{-1/2},2^{-1},2^{-3/2},2^{-2},...\right)\right\| = \sum_{k=1}^{\infty}\frac{1}{2^k} = 1$$

and

$$\left\|B\left(0,2^{-1/2},2^{-1},2^{-3/2},2^{-2},...\right)\right\| = \sum_{k=1}^{\infty}\frac{1}{2^k} = 1$$

so that B is also norm attaining with $\|B\| = 1$.

Example. Recall that the forward shift $F : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ is defined by

$$F(a_1,a_2,a_3,...) = (0,a_1,a_2,a_3,...)$$

and notice that

$$\|(a_1,a_2,a_3,...)\| = \|F(a_1,a_2,a_3,...)\|$$

and

$$\frac{\|F(a_1,a_2,a_3,...)\|}{\|(a_1,a_2,a_3,...)\|} = 1$$

as long as $(a_1,a_2,a_3,...) \neq 0$. Choosing the same sequence as in the previous example, we see that $\|F\| = 1$ and that F is norm attaining.

Example. Let $X = L^{\infty}([0,1])$. Define $T(f) = \int_0^1 f(x)\,dx$. By the properties of the integral, it is clear that T is linear. To compute the norm of T consider a unit vector f: this means that $\sup|f(x)| = 1$. Then $\|T\| = \int_0^1 1\ dx = 1$, and we have also shown that T is norm attaining.

Example. Let $X = C^1([0,1])$, the set of continuously differentiable functions on the unit interval, under $\|\cdot\|_{\infty}$. Define $T(f) = \frac{d}{dx}(f)$. By the properties of the derivative, T is linear. For $\alpha > 0$, consider the function $f_{\alpha}(x) = (1+\alpha)^{-1/2}(x+\alpha)^{1/2}$. It is easy to see that $\|f_{\alpha}\|_{\infty} = 1$, but $\|T(f_{\alpha})\|_{\infty} = \left\|\frac{1}{2}(1+\alpha)^{-1/2}(x+\alpha)^{-1/2}\right\|_{\infty} = \frac{1}{2}(1+\alpha)^{-1/2}\alpha^{-1/2}$ can be made arbitrarily large by choosing α to be small. Hence T is not a bounded operator, since $\|T\| \geq \frac{1}{2}(1+\alpha)^{-1/2}\alpha^{-1/2}$ for all $\alpha > 0$, and we must have that $\|T\| = \infty$. We know that T could not possibly be norm-attaining, as no function f could possibly have a continuous derivative such that $f'(x_0) = \infty$.

To continue our discussion of linear operators, we will discuss some properties of bounded linear operators.

Theorem 31 *Let $T : X \to Y$ be a linear map. Then the following statements are equivalent:*

1. *T is bounded.*
2. *T is continuous on X.*
3. *T is continuous at x_0 in X.*
4. *T is Lipschitz continuous on X.*

Proof. We show (1) implies (2), (3), and (4). Assume $\|T\|$ is bounded. Then

$$\infty > \|T\| = \sup_{x \in X - \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X} \geq \frac{\|T(x)\|_Y}{\|x\|_X}$$

as long as x is nonzero. Thus

$$\|T(x)\|_Y \leq \|T\|\,\|x\|_X$$

Hence we have that

$$\|T(x) - T(y)\|_Y = \|T(x-y)\|_Y \leq \|T\|\,\|x-y\|_X$$

and so T is Lipschitz continuous. We proceed by showing (3) implies (1). Assume T is continuous at $0 \in X$. Note that

$$T\left(\vec{0}\right) = T\left(0 \cdot \vec{0}\right) = 0T\left(\vec{0}\right) = 0$$

Since T is continuous at $x = 0$, for $\epsilon > 0$, we can find a δ so that whenever $0 < \|x\|_X \leq \delta$,

$$\left\|T(x) - T\left(\vec{0}\right)\right\|_Y = \|T(x)\|_Y < \epsilon$$

so, in particular, we can pick a δ_0 so that $\|T(x)\|_Y < 1$ whenever $0 < \|x\|_X \leq \delta_0$. Fix any nonzero $x \in X$ and let $y = \frac{\delta_0 x}{\|x\|_X}$. Then we have $\|y\|_X = \frac{\delta_0 \|x\|_X}{\|x\|_X} = \delta_0$. Now,

$$1 \geq \|T(y)\| = \left\|T\left(\frac{\delta_0 x}{\|x\|_X}\right)\right\| = \frac{\delta_0}{\|x\|_X} \|T(x)\|$$

and so

$$T(x) \leq \frac{1}{\delta_0} \|x\|_X$$

So, since $\frac{1}{\delta}$ is independent of our choice of x,

$$\sup_{x \in X - \{0\}} \frac{T(x)}{\|x\|_X} \leq \frac{1}{\delta_0}$$

and we have verified that T is bounded since δ_0 is some fixed positive number. If T is continuous at an arbitrary point x_0, then

$$\|T(x) - T(x_0)\| = \|T(x - x_0) - T(0)\|$$

and, letting $u = x - x_0$, the argument from above holds. The same argument shows that (2) and (4) also impliy (1), which completes the proof. ∎

Definition 32 *We will denote by $\mathcal{B}(X, Y)$ the set of all bounded linear operators from X to Y. If $X = Y$, we will abbreviate the notation and use $\mathcal{B}(X)$ instead. In the case where $Y = \mathbb{C}$, we will write $\mathcal{B}(X, Y) = X^*$ and call X^* the* dual space *(or simply the* dual*) of X.*

Remark. While the definition of the dual space may at first seem trivial, they are actually more familiar than one might expect. When $1 < p < \infty$, and q such that $\frac{1}{p} + \frac{1}{q} = 1$, the dual space of $L^p(X, d\mu)$ is $L^q(X, d\mu)$, and $\left((L^p)^*\right)^* = L^p$. Also, $\left(L^1(X, d\mu)\right)^* = L^\infty(X, d\mu)$[5]. As a result, $(\ell^p)^* = \ell^q$ and $\left(\ell^1\right)^* = \ell^\infty$. As one might expect, these results depend heavily on measure theory; the interested reader is referred to [1]. That being said, properties of dual spaces tell us something about the properties of the L^p spaces.

Theorem 33 $\mathcal{B}(X, Y)$ *is a Banach space whenever* Y *is a Banach space.*

Proof. We will not show that $\mathcal{B}(X, Y)$ is a normed linear space since it is quite elementary to do so; the crux of the matter is the completeness of $\mathcal{B}(X, Y)$, and [4, p. 98] contains a simple argument which we will present here. Consider a Cauchy sequence $\{T_n\}_{n=1}^\infty$ in $\mathcal{B}(X, Y)$. Then the sequence $\{T_n(x)\}_{n=1}^\infty = \{y_n\}_{n=1}^\infty$ is a subset of Y and since T_k is a bounded linear operator, we know that $\{y_n\}_{n=1}^\infty$ is Cauchy in Y and $y_n \to y \in Y$. Thus we have established that $T_n(x)$ converges pointwise in Y, and we can now define

$$T(x) = \lim T_n(x)$$

where T is, by the properties of the T_n, a bounded linear operator:

$$\begin{aligned} T(\alpha x - \beta y) &= \lim T_n(\alpha x - \beta y) \\ &= \lim\left[\alpha T_n(x) - \beta T_n(y)\right] \\ &= \alpha \lim T_n(x) - \beta \lim T_n(y) \\ &= \alpha T(x) - \beta T(y) \end{aligned}$$

Note that $T(x)$ is bounded because any Cauchy sequence has a uniform upper bound, and so

$$\|T_n\|_{\mathcal{B}(X,Y)} \leq A\|x\|_X$$

[5]However, $\left(\left(L^1(X, d\mu)\right)^*\right)^* \neq L^1(X, d\mu)$ - it turns out that the dual of $L^\infty(X, d\mu)$ is larger than $L^1(X, d\mu)$.

implies that

$$\|T\| \leq A \|x\|_X$$

Since $\{T_n\}_{n=1}^{\infty}$ is a Cauchy sequence, given any $\epsilon > 0$ we can find an N such that $m \geq n > N$ implies $\|T_m - T_n\|_{\mathcal{B}(X,Y)} < \epsilon$. So, in particular, it must hold for all nonzero x that

$$\begin{aligned}\left\|T_m\left(\frac{x}{\|x\|_X}\right) - T_n\left(\frac{x}{\|x\|_X}\right)\right\|_{\mathcal{B}(X,Y)} &\leq \sup_{x\in X-\{0\}}\left\|T_m\left(\frac{x}{\|x\|_X}\right) - T_n\left(\frac{x}{\|x\|_X}\right)\right\|_{\mathcal{B}(X,Y)} \\ &= \|T_m - T_n\|_{\mathcal{B}(X,Y)} < \epsilon\end{aligned}$$

and so

$$\|T_m(x) - T_n(x)\|_{\mathcal{B}(X,Y)} < \|x\|_X \epsilon$$

Notice that this result holds for any $m > n$, and so it must also hold in the limit, relaxing the strict inequality:

$$\lim_{m\to\infty} \|T_m(x) - T_n(x)\|_{\mathcal{B}(X,Y)} = \|T(x) - T_n(x)\|_{\mathcal{B}(X,Y)} \leq \|x\|_X \epsilon$$

and so

$$\left\|T\left(\frac{x}{\|x\|_X}\right) - T_n\left(\frac{x}{\|x\|_X}\right)\right\|_{\mathcal{B}(X,Y)} \leq \epsilon$$

or,

$$\|T(x) - T_n(x)\|_{\mathcal{B}(X,Y)} \leq \epsilon$$

for all $\|x\| = 1$. Since this holds for all unit vectors, it must hold for the supremum, and

$$\sup_{\|x\|_X=1} \|T(x) - T_n(x)\|_{\mathcal{B}(X,Y)} = \|T - T_n\|_{\mathcal{B}(X,Y)} \leq \epsilon$$

which shows that $\{T_n\}_{n=1}^{\infty}$ converges in norm to T, and hence $\mathcal{B}(X,Y)$ is complete whenever Y is complete. ■

Remark. The above theorem is fairly surprising at first, since the convergence of a Cauchy sequence of operators in $\mathcal{B}(X,Y)$ depends only on the completeness of the space Y. After careful thought, though, the apparent miracle is not all that unlikely: the operators map elements to Y, so much of their character is dependent only on the space Y. That is to say that the operators, in some sense, "live" in Y. This theorem also establishes an interesting fact about dual spaces: they are always Banach spaces, since $\mathbb{C}$ is complete. On Hilbert spaces, there is an interesting relationship between $\mathcal{H}$ and $\mathcal{H}^*$. We shall state the following theorem; the proof is mostly technical and is available in [3, p. 43].

Theorem 34 *(Riesz Lemma). Let $\mathcal{H}$ be a Hilbert space. For each T in $\mathcal{H}^*$ there is a unique y_T in $\mathcal{H}$ such that $T(x) = (y_T, x)$ for all x in $\mathcal{H}$. Moreover, $\|y_T\|_{\mathcal{H}} = \|T\|_{\mathcal{H}^*}$*

Remark. It is of note that this theorem implies that any bounded linear functional on a Hilbert space is, simply put, the projection in the direction of a particular vector y_T composed with a stretching factor.

Definition 35 *Given a Hilbert space $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$, we define T^*, called the* adjoint of T, *to be a linear operator such that*

$$(Tx, y) = (x, T^*y)$$

for all $x, y \in \mathcal{H}$. If $T = T^*$, then we say T is self-adjoint *or that T is* Hermitian*.*

Remark. It is not difficult to prove that the operation of taking the adjoint, like the inverse, is involutive, that is $(A^*)^* = A$ and $(AB)^* = B^*A^*$. It is also the case that $(A^*)^{-1} = \left(A^{-1}\right)^*$ whenever A is invertible. These results follow immediately from the definition.

Theorem 36 *Given a Hilbert space $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$, the adjoint T^* is unique.*

Proof. Let A and B satisfy

$$\begin{aligned}(Tx, y) &= (x, Ay)\\(Tx, y) &= (x, By)\end{aligned}$$

for all x and y in $\mathcal{H}$. Then it must be that

$$\begin{aligned}(Tx,y)-(Tx,y) &= (x,Ay)-(x,By)\\ 0 &= (x,(A-B)\,y)\end{aligned}$$

which means that $(A-B)\,y$ is normal to every element of $\mathcal{H}$. Thus $(A-B)\,y=0$, which implies that $Ay=By$ for all y, so $A=B$. ■

Proposition 37 *If $T\in\mathcal{B}(\mathcal{H})$ then $\|T\|=\|T^*\|=\|T^*T\|^{1/2}$.*

Proof. Let $h\in\mathcal{H}$ such that $\|h\|\leq 1$. Then

$$\begin{aligned}\|Ah\|^2 &= (Ah,Ah)=(A^*Ah,h)\leq\|A^*Ah\|\,\|h\|\\ &\leq \|A^*Ah\|\leq\|A^*A\|\end{aligned}$$

Now, for any A,B in $\mathcal{B}(\mathcal{H})$ and for any unit vector h in $\mathcal{H}$, we must have that $\|ABh\|\leq\|A\|\,\|Bh\|\leq\|A\|\,\|B\|$. Thus

$$\|Ah\|^2\leq\|A^*A\|\leq\|A\|\,\|A^*\|$$

and so

$$\begin{aligned}\|A\|^2 &\leq \|A\|\,\|A^*\|\\ \|A\| &\leq \|A^*\|\end{aligned}$$

Applying the same argument, we have that

$$\begin{aligned}\|A^*h\|^2 &= (A^*h,A^*h)=(A^*A^{**}h,h)\leq\|A^*A^{**}h\|\,\|h\|\\ &\leq \|A^*A^{**}h\|\leq\|A^*A^{**}\|\leq\|A^*\|\,\|A^{**}\|\end{aligned}$$

and so

$$\begin{aligned}\|A^*\|^2 &\leq \|A^*\|\,\|A^{**}\|\\ \|A^*\| &\leq \|A^{**}\|=\|A\|\end{aligned}$$

Thus

$$\|A\|^2 \leq \|A^*A\| \leq \|A\| \|A^*\| = \|A\|^2$$

and we have that $\|A\| = \|A^*\|$ and $\|A^*A\| = \|A\|^2$. ∎

Remark. This theorem may at first seem somewhat unhelpful, but it can be the case that computing the norm of the adjoint is significantly easier than determining the norm of the actual operator or, in some cases, $\|T^*T\|$ is the easiest to calculate of the three.

Proposition 38 *If A is a Hermitian operator, then (Ax, x) is always a real number.*[6]

Proof.

$$(Ax, x) = \overline{(x, Ax)}$$

by the properties of inner products, but we also have that

$$(Ax, x) = (x, Ax)$$

and thus

$$\overline{(x, Ax)} = (x, Ax)$$

and so (Ax, x) is always a real number. ∎

We shall continue with our discussion of operators by generalizing the notion of eigenvector.

Definition 39 *An* algebra *is a vector space $\mathcal{A}$ together with a multiplicative binary operation that makes $\mathcal{A}$ a ring such that*

$$\alpha\,(xy) = (\alpha x)\,y = x\,(\alpha y)$$

[6] Most quantities (particularly dynamic quantities such as momentum) in quantum mechanics are represented as operators. The momentum operator is $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. On the set of solutions to a Schrödinger Equation, this operator (and, in fact, all quantum mechanical operators corresponding to observables) is Hermitian and so the expectation value $\langle \hat{p} \rangle = (\psi, \hat{p}\psi) = \int \overline{\psi}\hat{p}\psi d\tau$ is a real quantity. While this may not seem to be of such huge mathematical importance, it is critical physically: the momentum operator must yield real eigenvectors and expectation values or else it would be physically meaningless

for all x, y in $\mathcal{A}$ and scalars α. A Banach algebra *is a space $\mathcal{A}$ with the above properties with the added requirement that*

$$\|xy\| \leq \|x\| \, \|y\|$$

for all x, y in $\mathcal{A}$. If $\mathcal{A}$ has an identity, we will denote it as e and we will assume $\|e\| = 1$.

Definition 40 *We say an operator T is* invertible *if there exists an operator T^{-1} such that $TT^{-1} = T^{-1}T = I$ where I is the identity operator.*[7] *An* eigenvector *of an operator T is a nonzero vector x so that $Tx = \lambda x$ where λ is a complex number (λ is called the* eigenvalue corresponding to *x).*

Definition 41 *We define the* spectrum of *T to be $\sigma(T) = \{t \in \mathbb{C} : T - t$ is not invertible$\}$. The* resolvent set *is $\rho(T) = \mathbb{C} \setminus \sigma(T)$. If λ is an eigenvalue of T, then it is clear that λ is in $\sigma(T)$. The set of all eigenvalues of T is referred to as the* point spectrum of *T. The set of all elements of the spectrum of T which are not eigenvalues is called the* residual spectrum of *T.*

Example. Consider a continuous function $f : [0,1] \to \mathbb{R} \in C([0,1])$. Then if $x_0 = f(x)$ for some x, it follows that $f(x) - x_0$ has a zero in $[0,1]$. Therefore $f(x) - x_0$ is not invertible, and x_0 is in $\sigma(f)$. If $x_0 \neq f(x)$ for any x in $[0,1]$, then $(f(x) - x_0)^{-1}$ is well defined and f is invertible, so $x_0 \notin \sigma(f)$. Hence $\sigma(f) = f([0,1])$.

Proposition 42 *Let X be a Banach space. Suppose that $T \in \mathcal{B}(X)$ with $\|T\| < 1$. Then $I - T$ is invertible in $\mathcal{B}(X)$ and its inverse is given by*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

[7] In finite dimensional vector spaces, the requirement that $TT^{-1} = T^{-1}T = I$ is redundant because, as one can show, if $T^{-1}T = I$ then $TT^{-1} = I$. However, as the left and right shift operators show on the sequence of square summable sequences, it is the case that $BF = I$ (since shifting forward and then backward does not change a sequence) , but $FB \neq I$ in general (since goinf backward and then forward makes the first entry zero), and so the requirement that $TT^{-1} = T^{-1}T$ is non-trivial in the infinite-dimensional case.

Proof. We shall adapt a proof from [4, p. 100]. First, note that

$$\begin{aligned}\left\|T^k x\right\| &= \left\|T\left(T^{k-1}x\right)\right\| \leq \|T\| \left\|T^{k-1}x\right\| \leq \|T\|^2 \left\|T^{k-2}x\right\| \\ &\leq \ldots \leq \|T\|^{k-1} \|Tx\| \leq \|T\|^k\end{aligned}$$

and so

$$\left\|T^k\right\| \leq \|T\|^k$$

So we have that

$$\sum_{k=0}^{\infty} \left\|T^k\right\| \leq \sum_{k=0}^{\infty} \|T\|^k = \frac{1}{1-\|T\|}$$

Recall that in chapter 1, we proved that if X is a complete vector space, $\sum_{k=1}^{\infty} f_k$ converges in X (in norm) whenever $\sum_{k=1}^{\infty} \|f_k\|$ converges. We proved above that $\mathcal{B}(X)$ is complete, so $\sum_{k=1}^{\infty} T^k$ must be an element of $\mathcal{B}(X)$. Now, we know that T and $I-T$ are bounded linear operators and are thus continuous, so

$$\begin{aligned}\left((I-T)\sum_{k=0}^{\infty} T^k\right)x &= \left((I-T)\lim_{n\to\infty}\sum_{k=0}^{n} T^k\right)x \\ &= \left(\lim_{n\to\infty}\sum_{k=0}^{n}(I-T)T^k\right)x \\ &= \lim_{n\to\infty}\left(x - T^{n+1}x\right)\end{aligned}$$

where this last step must hold since

$$\begin{aligned}\sum_{k=0}^{n}(I-T)T^k &= I - T + T - T^2 + T^2 - T^3 + \ldots - T^n + T^n - T^{n+1} \\ &= I - T^{n+1}\end{aligned}$$

Continuing, we have

$$\begin{aligned}\left((I-T)\sum_{k=0}^{\infty} T^k\right)x &= \lim_{n\to\infty}\left(x - T^{n+1}x\right) \\ &= x - \lim_{n\to\infty} T^{n+1}x\end{aligned}$$

By our previous work, $\|T^n x\| \leq \|T\|^n$, so since $\|T\| < 1$,

$$\begin{aligned} \lim_{n\to\infty} \left\|T^{n+1}x\right\| &\leq \lim_{n\to\infty} \left\|T^{n+1}\right\| \\ &= 0 \end{aligned}$$

and so $\lim\limits_{n\to\infty} T^{n+1}x = 0$. Thus

$$\left((I-T)\sum_{k=0}^{\infty} T^k\right) x = x$$

This analysis holds for

$$\left(\left(\sum_{k=0}^{\infty} T^k\right)(I-T)\right) x$$

and so $(I-T)^{-1} = \sum\limits_{k=0}^{\infty} T^k$. ■

Corollary 43 *If S, T are in $\mathcal{B}(X)$ where X is a Banach space, T is invertible in $\mathcal{B}(X)$, and $\|T-S\| < \frac{1}{\|T^{-1}\|}$, then S is invertible in $\mathcal{B}(X)$.*

Proof. Again, we appeal to an argument from [4, p. 101]. Since $\|T-S\| < \frac{1}{\|T^{-1}\|}$, we have that $\left\|T^{-1}\right\| \|T-S\| < 1$, and so the above theorem holds for $I - ST^{-1}$ since

$$\left\|(T-S)\,T^{-1}\right\| \leq \left\|T^{-1}\right\| \|T-S\| < 1$$

Thus $I - \left(I - ST^{-1}\right) = ST^{-1}$ is invertible. Since the product of two invertible operators is invertible (the inverse is an involution), we have it that

$$\left(ST^{-1}\right) T = S$$

is invertible. ■

Theorem 44 *Let X be a Banach space. The spectrum of every T in $\mathcal{B}(X)$ is a compact and nonempty set and is a subset of $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$.*

Proof. (from [4, p. 103]). We are not able to prove that $\sigma(T)$ is non-empty without the use of some deep theorems from analytic function theory, so this aspect shall remain unproved. To prove that $\sigma(T)$ is compact, we shall prove that it is closed and bounded. To prove that $\sigma(T)$ is closed, we show that its resolvent set is open. Suppose $\lambda \in \rho(T)$. Then $(\lambda I - T)^{-1}$ exists. Now, suppose that $|\lambda - \mu| < \left\| (\lambda I - T)^{-1} \right\|^{-1}$. Since

$$|\lambda - \mu| = \|(\lambda - \mu) I\| = \|(\lambda I - T) - (\mu I - T)\|$$

the previous theorem implies that $\mu I - T$ must be invertible and so μ is in $\rho(T)$. Then any sufficiently small ball around λ must be contained in $\rho(T)$ and so $\rho(T)$ is open. Thus $\sigma(T)$ must be closed. If $|\lambda| > \|T\|$, then $\left\|\frac{T}{\lambda}\right\| < 1$ and $\left(I - \frac{T}{\lambda}\right)^{-1}$ is well-defined, and so $\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = (\lambda I - T)^{-1}$ is also well-defined. Hence $\lambda \in \rho(T)$ and $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$. This shows that $\sigma(T)$ is closed and bounded. It follows from the Heine–Borel that $\sigma(T)$ is compact. ■

Definition 45 *The* spectral radius of $T \in \mathcal{B}(X)$ *is given by*

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

Remark. By our previous work, it is worth noting that $r(T) \leq \|T\|$. It is also worth noting that if the largest element (should one exist) of $\sigma(T)$ is an eigenvalue of T, then there is an x in X such that $T(x) = \lambda x$ and so

$$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \left\| \lambda \frac{x}{\|x\|} \right\| = \lambda = r(T)$$

which implies that $\|T\| \leq \lambda$, by taking the supremum over all x in X. Hence we have that $\|T\| \leq r(T) \leq \|T\|$, and hence $r(T) = \|T\|$. So, being able to compute the spectral radius can be a useful tool to compute the norm of an operator in $\mathcal{B}(X)$. At the very worst, we are able to provide a lower bound for $\|T\|$.

Theorem 46

$$r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}$$

Proof. We cannot give a complete proof without reference to results beyond the scope of this document, so the interested reader is directed to [4, p. 105]. ■

Proposition 47 *Let $\mathcal{H}$ be a Hilbert space and let T be a Hermitian operator in $\mathcal{B}(\mathcal{H})$. Then $r(T) = \|H\|$.*

Proof. (from [4, p. 115]). Recall that $\|T^n\| \leq \|T\|^n$ for all elements of $\mathcal{B}(\mathcal{H})$. Also note that

$$\|Tx\|^2 = (Tx, Tx) = \left(T^2x, x\right) \leq \left\|T^2x\right\| \|x\| \leq \left\|T^2\right\| \|x\|^2$$

and assuming $x \neq 0$,

$$\frac{\|Tx\|^2}{\|x\|^2} \leq \left\|T^2\right\|$$

By taking the supremum over all x, we have that $\|T\|^2 \leq \left\|T^2\right\|$ and thus $\|T\|^2 = \left\|T^2\right\|$. Likewise,

$$\left\|T^k x\right\|^2 = \left(T^k x, T^k x\right) = \left(T^{2k}x, x\right) \leq \left\|T^{2k}x\right\| \|x\|^2 \leq \left\|T^{2k}\right\| \|x\|^2$$

and by the same reasoning as above, $\left\|T^k\right\|^2 \leq \left\|T^{2k}\right\|$. Induction (let $k = 2^p$) can now guarantee that

$$\|T\|^{2^m} = \left\|T^{2^m}\right\|$$

Now, let $1 \leq n \leq 2^m$. Then

$$\begin{aligned}
\left\|T^{2^m}\right\| &= \left\|T^n T^{2^m - n}\right\| \\
&\leq \|T^n\| \left\|T^{2^m-n}\right\| \\
&\leq \|T^n\| \|T\|^{2^m-n} \\
&\leq \|T\|^n \|T\|^{2^m-n} \\
&= \|T\|^{2^m} = \left\|T^{2^m}\right\|
\end{aligned}$$

and thus

$$\|T^n\| \|T\|^{2^m-n} = \left\|T^{2^m}\right\|$$

and so

$$\begin{aligned}
\|T^n\| \|T\|^{-n} &= 1 \\
\|T^n\| &= \|T\|^n
\end{aligned}$$

Thus

$$\begin{aligned} r(T) &= \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left(\|T\|^n\right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|T\| \\ &= \|T\| \end{aligned}$$

whence $r(T) = \|T\|$. ■

This discussion concludes our survey of functional analysis. The following section will provide an application of functional analysis to the field of partial differential equations and give some evidence as to why functional analysis has become one of the fundamental fields of modern mathematics.

Chapter 4.

Theorem 48 *(Contraction Mapping Fixed Point Theorem). Let X be a Banach space. Any map $T : X \to X$ such that for all $x, y \in X$ we have the condition that $\|Tx - Ty\| \leq C \|x - y\|$ where $0 \leq C < 1$ is called a contraction map. (Note that this means T is Lipschitz continuous with coefficient $C < 1$.) Any contraction map has a fixed point and this fixed point is unique.*

Proof. Let $T : X \to X$ be a contraction mapping on a Banach space X. Fix some vector x_0 in X, and define a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $x_n = T(x_{n-1})$ for $n \geq 1$. If any $x_n = x_{n-1}$, then we are done, since $T(x_{n-1}) = x_n = x_{n-1}$. So, assume that $x_n - x_{n-1} \neq 0$ for any n. Since T is a contraction map,

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T(x_n) - T(x_{n-1})\| \leq C \|x_n - x_{n-1}\| \\ &= C \|T(x_{n-1}) - T(x_{n-2})\| \leq C^2 \|x_{n-1} - x_{n-2}\| \\ &\leq C^3 \|x_{n-2} - x_{n-3}\| \leq \ldots \leq C^n \|x_1 - x_0\| \end{aligned}$$

So, making use of the triangle inequality for any norm, if $m > n$, we have that

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{k=n+1}^{m} \|x_k - x_{k-1}\| \leq \sum_{k=n+1}^{m} C^{k-1} \|x_1 - x_0\| \\ &= \frac{C^n (1 - C^m)}{1 - C} \|x_1 - x_0\| \leq \frac{C^n}{1 - C} \|x_1 - x_0\| \end{aligned}$$

If, for any $\epsilon > 0$, we choose n large enough that $C^n < \epsilon \frac{1-C}{\|x_1 - x_0\|}$, then $\|x_m - x_n\| < \epsilon$ which proves $\{x_n\}_{n=0}^{\infty}$ is Cauchy and, hence, converges by the completeness property of our Banach space. Thus $\lim x_n = x \in X$. Since T is Lipschitz continuous, it is continuous, and so

$$x = \lim x_n = \lim T(x_{n-1}) = T(\lim x_{n-1}) = T(x)$$

and so x is indeed a fixed point of T. To show that this is the only fixed point, let x and y be fixed points of T. Then

$$\|x - y\| = \|T(x) - T(y)\| \leq K \|x - y\|$$

Since $K < 1$, we must conclude that $\|x - y\| = 0$, i.e. that $x = y$. ■

Definition 49 *We define a set A to be* convex *if for every $x, y \in A$ and every $0 \leq \lambda \leq 1$, the sum $\lambda x + (1 - \lambda) y \in A$. Geometrically, this property guarantees that given any two points in A, every point on the line segment between them is also in A.*

Theorem 50 *Theorem (Generalized Brouwer Fixed Point Theorem[8]). Let C be a compact, convex subset of $\mathbb{R}^n$ and suppose $f : C \to C$ is continuous. Then f has a fixed point.*

Proof. See [3, p. 364]. ■

Theorem 51 *(Schauder Fixed Point Theorem). If $K \subset B$ is a compact, convex subset of a Banach space B, and $T : K \to K$ is continuous, then T has a fixed point.*

Proof. Let $\epsilon > 0$ be given. Cover K by $B_\epsilon(x)$ for all x in K where $B_\epsilon(x) = \{y \in K : \|y - x\| < \epsilon\}$. Since K is compact, there exists a finite subcover and so for some finite N, we have that $K \subset \bigcup_{j=1}^{N} B_\epsilon(x_j)$. Now, define the function $h_j(x) = \max\left\{\frac{\epsilon - \|x - x_j\|}{\epsilon}, 0\right\}$. It is clear that $h_j(x) > 0$ for some j since x has to belong to some $B_\epsilon(x_j)$. Define

$$T_\epsilon(x) = \frac{\sum_{j=1}^{N} h_j(x) T(x_j)}{\sum_{j=1}^{N} h_j(x)}$$

This is a well-defined function, since $\sum_{j=1}^{N} h_j(x) > 0$. Now, $T_\epsilon : K \to C_\epsilon$ where C_ϵ is the convex hull of $\{F(x_1), ..., F(x_N)\}$, a set which is finite dimensional. Note that C_ϵ is a subset of K. So, if we restrict the domain to C_ϵ, i.e. consider $T_\epsilon : C_\epsilon \to C_\epsilon$, then we can apply the generalization of the Brouwer Fixed Point Theorem to convex sets and guarantee ourselves an $x_\epsilon \in C_\epsilon$ such that $T_\epsilon(x_\epsilon) = x_\epsilon$. Let $\epsilon_k \to 0$ and choose x_{ϵ_k} to be a sequence of fixed points in the process we defined

[8] We call this the generalized Brouwer Fixed Point Theorem because the classic theorem is proven only for functions on the unit ball (see [2]), however the authors in [3] refer to the generalization as the Brouwer Fixed Point Theorem.

above. K is compact, so $x_{\epsilon_k} \to x_0 \in K$. Therefore

$$\|T(x_0) - x_0\| \leq \|T(x_0) - T(x_{\epsilon_k})\| + \|T(x_{\epsilon_k}) - T_{\epsilon_k}(x_{\epsilon_k})\| + \|T_{\epsilon_k}(x_{\epsilon_k}) - x_{\epsilon_k}\| + \|x_{\epsilon_k} - x_0\|$$

The quantity $\|T_{\epsilon_k}(x_{\epsilon_k}) - x_{\epsilon_k}\| = 0$ since x_{ϵ_k} is a fixed point of T_{ϵ_k}. Also, by choosing ϵ_k is very small, $\|x_{\epsilon_k} - x_0\| < \epsilon$. Since T is continuous, we can choose ϵ_k to be very small and guarantee $\|T(x_0) - T(x_{\epsilon_k})\| < \epsilon$. Now,

$$\begin{aligned}
\|T(x_{\epsilon_k}) - T_{\epsilon_k}(x_{\epsilon_k})\| &= \left\| T(x_{\epsilon_k}) - \frac{\sum_{j=1}^{N} h_j(x_{\epsilon_k})\, T(x_j)}{\sum_{j=1}^{N} h_j(x_{\epsilon_k})} \right\| \\
&= \left\| \frac{\sum_{j=1}^{N} h_j(x_{\epsilon_k})\, [T(x_{\epsilon_k}) - T(x_j)]}{\sum_{j=1}^{N} h_j(x_{\epsilon_k})} \right\|
\end{aligned}$$

Unless x_{ϵ_k} is such that $\|x_{\epsilon_k} - x_j\| < \epsilon$, we have that $h_j(x_{\epsilon_k}) = 0$. When $h_j(x_{\epsilon_k}) \neq 0$, we still have that $T(x_{\epsilon_k}) - T(x_j)$ is very small since T is continuous, and so the entire quantity in the last line above must be very small, hence $\|T(x_{\epsilon_k}) - T_{\epsilon_k}(x_{\epsilon_k})\| < \epsilon$. So,

$$\begin{aligned}
\|T(x_0) - x_0\| &\leq \|T(x_0) - T(x_{\epsilon_k})\| + \|T(x_{\epsilon_k}) - T_{\epsilon_k}(x_{\epsilon_k})\| + \|T_{\epsilon_k}(x_{\epsilon_k}) - x_{\epsilon_k}\| + \|x_{\epsilon_k} - x_0\| \\
&< \epsilon + \epsilon + 0 + \epsilon = 3\epsilon
\end{aligned}$$

Since ϵ is aribitrary, it must be the case that $T(x_0) = x_0$ and hence x_0 is a fixed point of T. ■

Remark. These two fixed point theorems are not created equal. Most operators which have fixed points have more than one, but the contraction maps are a class of operators which do have unique fixed points — something which could be quite useful in proving other theorems. The Schauder theorem is highly general and therefore significantly more powerful than the contraction mapping theorem which is limited to a relatively small class of operators. That being said, fixed points for contraction maps are very easy to approximate by numerical methods — simply choose any point in the Banach space and apply the operator enough times that the difference between iterations is very small. For Schauder's theorem, there is no general numerical approximation algorithm. As we shall see in the following example, these fixed point theorems can be used to solve partial differential equations. These solutions are of incredible scientific interest, and so numerical solutions are highly desirable.

Consider the following autonomous partial differential equation:

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ have continuous second-order derivatives and let $f(u)$ be continuous in u. Then, given the boundary data $u(x) = \varphi$ on $\partial\Omega$ we wish to study the PDE

$$\Delta u = f(u)$$

There is no general method for solving PDEs directly. There is, however, a systematic approach in which solutions can be proven to exist. The definition of solution is relaxed to what is called a "weak solution" in such a way that proving existence of these weak solutions is more easily approached and that if these weak solutions are continuously differentiable they are actual solutions (called "classical solutions").

Definition 52 *A function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a* weak solution *of $\Delta u = f(u)$ on domain $\Omega \subset \mathbb{R}^n$ if for every function $\zeta \in C_c^\infty(\Omega)$, the set of infinitely differentiable functions with compact support in Ω, i.e. ζ and all its derivatives are identically zero on $\partial\Omega$, we have*

$$\int_\Omega u\Delta\zeta = \int_\Omega f(u)\,\zeta$$

Remark. Note that if $u \in C^1(\Omega)$, then we can integrate by parts to get

$$\begin{aligned}\int_\Omega u\Delta\zeta &= \int_{\partial\Omega} u\nabla\zeta \cdot dS - \int_\Omega \nabla u \cdot \nabla\zeta \\ &= -\int_\Omega \nabla u \cdot \nabla\zeta\end{aligned}$$

since $\nabla\zeta$ is zero in $\partial\Omega$. By a similar argument, if $u \in C^2(\Omega)$, then

$$\int_\Omega u\Delta\zeta = -\int_\Omega \nabla u \cdot \nabla\zeta = \int_\Omega \zeta\Delta u$$

and

$$\int_\Omega u\Delta\zeta = \int_\Omega \zeta\Delta u = \int_\Omega \zeta f(u)$$

Since ζ is essentially arbitrary, it is straightforward from here to deduce that if u is a weak solution and u has continuous second-order partial derivatives, then $\Delta u = f(u)$. Thus we see that our definition of weak solution is simply a generalized version of a solution to the PDE $\Delta u = f(u)$.

Weak solutions also allow the study of PDEs in a non-continuous sense. It is not always the case that we are interested in highly smooth solutions. Indeed, in physical applications one may be interested in non-differentiable or perhaps even non-continuous solutions: studying the current in a circuit when turning on a light switch, for example, provides a discontinuous jump in voltage and the solution to a differential equation modeling such a system will not be differentiable at the time when the switch is turned on. Likewise, in the study of solid state physics, one of the first PDEs studied is

$$\Delta\Psi + V \cdot \Psi = E \cdot \Psi$$

where E is the energy of a particular solution and V is a periodic Dirac δ function or a square wave. Clearly, finding a solution with continuous second-order derivatives to this equation is challenging!

We shall complete this section by studying a particularly simple non-linear differential equation and applying the Schauder Fixed-Point Theorem to prove the existence of solutions.

During an internship at Cornell University, the author, along with Phillip Whitman, Frances Hammock, and Alexander Meadows, studied the differential equation discussed below. Much of the analysis completed below is due in no small part to their efforts. Consider the PDE

$$\Delta u = u^{-\alpha}$$

for $0 < \alpha < 1$, where $u : B_\delta(0) \subset \mathbb{R}^n \to \mathbb{R}$. It turns out that for autonomous equations of this form, as long as the right hand side of the equation is continuous, classical solutions exist. This happens as long as we require $u(x) > 0$. So, we shall relax the condition and only require $u(x) \geq 0$. We shall also restrict ourselves to solutions which are radially symmetric, i.e. $u(x)$ is a function of the modulus of x alone, so that $u(0) = 0$ and $u_r(0) = 0$. In such a case, the PDE in question reduces to the ordinary differential equation

$$u_{rr}+\frac{n-1}{r}u_r=u^{-\alpha}$$

Slightly modifying this equation, we see that

$$u^{-\alpha}=u_{rr}+\frac{n-1}{r}u_r=r^{1-n}\left(r^{n-1}u_r\right)_r$$

Assuming the existence of a solution and by dividing by r^{1-n} and integrating both sides, we see that

$$\int_0^r t^{n-1}u\left(t\right)^{-\alpha}\,dt=r^{n-1}u_r$$

Dividing by r^{n-1} and integrating, we have

$$\int_0^r s^{1-n}\int_0^s t^{n-1}u\left(t\right)^{-\alpha}\,dtds=u$$

We see now that if we can compute a fixed point of

$$T\left(v\right)\left(r\right)=\int_0^r s^{1-n}\int_0^s t^{n-1}v\left(t\right)^{-\alpha}\,dtds$$

and show it is sufficiently differentiable, we have found our desired solution.

Now, let $X=C^0\left([0,\delta]\right)$ (under the sup norm), and take

$$K=\left\{v\in X:C_1r^{2/1+\alpha}\leq v\left(r\right)\leq C_2r^{2/1+\alpha}\right\}$$

where $0<C_1<C_2$. We have previously shown that X under the sup norm is a Banach space. To see that K is convex, let v_1 and v_2 be elements of K. Then for $0<\lambda<1$,

$$\begin{aligned}\lambda v_1+\left(1-\lambda\right)v_2 &\leq \lambda C_2r^{2/1+\alpha}+\left(1-\lambda\right)C_2r^{2/1+\alpha}\\ &= C_2r^{2/1+\alpha}\end{aligned}$$

and

$$\begin{aligned}\lambda v_1+\left(1-\lambda\right)v_2 &\geq \lambda C_1r^{2/1+\alpha}+\left(1-\lambda\right)C_1r^{2/1+\alpha}\\ &= C_1r^{2/1+\alpha}\end{aligned}$$

which implies that K is convex. To see that K is closed, consider sequence $\{v_n(r)\}$ which is convergent in X such that $v_n \in K$ for all n. Then

$$C_1 r^{2/1+\alpha} \leq v_n(r) \leq C_2 r^{2/1+\alpha}$$

and so

$$C_1 r^{2/1+\alpha} \leq \lim v_n(r) \leq C_2 r^{2/1+\alpha}$$

To see that K is bounded, consider any v in K. Then $\|v\|_\infty \leq \left\|C_2 r^{2/1+\alpha}\right\|_\infty = C_2 \sup\left\{r^{2/1+\alpha} : r \in [0,\delta]\right\} = C_2 \delta^{2/1+\alpha}$. So, K is closed and bounded, which implies it is compact. Hence all that remains is to show that T is continuous in v and r. Let v_1 and v_2 be elements of K. Let $\epsilon > 0$ be arbitrary.

$$\begin{aligned}
\|T(v_1)(r_1) - T(v_2)(r_2)\| &= \left\|\int_0^{r_1} s^{1-n} \int_0^s t^{n-1} v_1(t)^{-\alpha}\, dtds - \int_0^{r_2} s^{1-n} \int_0^s t^{n-1} v_2(t)^{-\alpha}\, dtds\right\| \\
&\leq \left\|\int_0^{r_1} s^{1-n} \int_0^s t^{n-1} \left[C_1 r^{2/1+\alpha}\right]^{-\alpha} dtds - \int_0^{r_2} s^{1-n} \int_0^s t^{n-1} \left[C_2 r^{2/1+\alpha}\right]^{-\alpha} dtds\right\|
\end{aligned}$$

which is true since $r^{2/1+\alpha}$ is always increasing and we are raising $v_1(t)$ and $v_2(t)$ to a negative power. Now,

$$\begin{aligned}
&\left\|\int_0^{r_1} s^{1-n} \int_0^s t^{n-1} \left[C_1 t^{2/1+\alpha}\right]^{-\alpha} dtds - \int_0^{r_2} s^{1-n} \int_0^s t^{n-1} \left[C_2 t^{2/1+\alpha}\right]^{-\alpha} dtds\right\| \\
= &\left\|\int_0^{r_1} s^{1-n} \int_0^s t^{n-1} \left[C_1 t^{2/1+\alpha}\right]^{-\alpha} dtds - \int_0^{r_2} s^{1-n} \int_0^s t^{n-1} \left[C_2 t^{2/1+\alpha}\right]^{-\alpha} dtds\right\| \\
= &\left\|C_1^{-\alpha} \int_0^{r_1} s^{1-n} \int_0^s t^{n-1} t^{-2\alpha/1+\alpha} dtds - C_2^{-\alpha} \int_0^{r_2} s^{1-n} \int_0^s t^{n-1} t^{-2\alpha/1+\alpha} dtds\right\| \\
= &\left\|C_1^{-\alpha} \int_0^{r_1} s^{1-n} \int_0^s t^{n-1-2\alpha/1+\alpha} dtds - C_2^{-\alpha} \int_0^{r_2} s^{1-n} \int_0^s t^{n-1-2\alpha/1+\alpha} dtds\right\|
\end{aligned}$$

Since $\alpha < 1$, it must be that $\frac{2\alpha}{1+\alpha} < 1$ and so, since $n \geq 1$, we see that $n - 1 - \frac{2\alpha}{1+\alpha} > -1$. Therefore the integration can be done via the power rule.

$$
\begin{aligned}
& \left\| C_1^{-\alpha} \int_0^{r_1} s^{1-n} \int_0^{s} t^{n-1-2\alpha/1+\alpha} dt ds - C_2^{-\alpha} \int_0^{r_2} s^{1-n} \int_0^{s} t^{n-1-2\alpha/1+\alpha} dt ds \right\| \\
= & \left\| \frac{C_1^{-\alpha}}{n-\frac{2\alpha}{1+\alpha}} \int_0^{r_1} s^{1-n} s^{n-2\alpha/1+\alpha} ds - \frac{C_2^{-\alpha}}{n-\frac{2\alpha}{1+\alpha}} \int_0^{r_2} s^{1-n} s^{n-2\alpha/1+\alpha} ds \right\| \\
= & \left\| \frac{C_1^{-\alpha}}{n-\frac{2\alpha}{1+\alpha}} \int_0^{r_1} s^{1-2\alpha/1+\alpha} ds - \frac{C_2^{-\alpha}}{n-\frac{2\alpha}{1+\alpha}} \int_0^{r_2} s^{1-2\alpha/1+\alpha} ds \right\| \\
= & \left\| \frac{C_1^{-\alpha}}{n-\frac{2\alpha}{1+\alpha}} \frac{r_1^{2-2\alpha/1+\alpha}}{2-\frac{2\alpha}{1+\alpha}} - \frac{C_2^{-\alpha}}{n-\frac{2\alpha}{1+\alpha}} \frac{r_2^{2-2\alpha/1+\alpha}}{2-\frac{2\alpha}{1+\alpha}} \right\| \\
= & \frac{(1+\alpha)^2}{2\left((1+\alpha)n-2\alpha\right)} \left\| C_1^{-\alpha} r_1^{2/1+\alpha} - C_2^{-\alpha} r_2^{2/1+\alpha} \right\|
\end{aligned}
$$

Certainly, as long as we require r_1 and r_2 to be close together, the quantity $\left\| C_1^{-\alpha} r_1^{2/1+\alpha} - C_2^{-\alpha} r_2^{2/1+\alpha} \right\|$ can be made as small as we should like, whence

$$\left\| T(v_1)(r_1) - T(v_2)(r_2) \right\| < \epsilon$$

for r_1 and r_2 close. Since T is continuous, the Schauder fixed point theorem guarantees that there is a u in K such that $T(u) = u$.

We have now guaranteed the existence of a weak radially symmetric solution to our PDE. In the following discussion, we shall prove that no solution (radially symmetric or otherwise) is $C^2([0,\delta])$.

Definition 53 *Let $0 \leq \beta < 1$. We say $u \in C^{k,\beta}(\Omega)$ if u has continuous kth order partial derivatives and is Hölder continuous with exponent β. By* Hölder continuity with exponent β, *we mean that for some positive constant C*

$$|u(x) - u(y)| \leq C |x-y|^{\beta} \quad \forall x, y \in \Omega$$

Note that the case where $\beta = 1$, i.e.

$$|u(x) - u(y)| \leq C |x-y| \quad \forall x, y \in \Omega$$

the function u is simply Lipschitz continuous.

Remark. The Hölder exponent could be considered a measure of "how differentiable" a function without continuous derivative is. We call the least upper bound for β the *regularity of* u provided u is not differentiable. If a function has maximum Hölder exponent $\frac{1}{2}$, the ratio $\frac{|u(x)-u(y)|}{|x-y|}$ is always smaller than $C|x-y|^{-\frac{1}{2}}$ and so as x and y get very close together, the restriction on how large $\frac{|u(x)-u(y)|}{|x-y|}$ could be increases faster than a function with Hölder exponent $\frac{2}{3}$, and so on, until β goes to one. So, larger values for β are more desirable (provided differentiability is a desirable property to begin with). For Lipschitz continuous functions, the ratio $\frac{|u(x)-u(y)|}{|x-y|}$ is always bounded; $\lim_{x\to y}\frac{|u(x)-u(y)|}{|x-y|}$ might still be undefined, but it is not diverging to $\pm\infty$, a property which can be quite useful.

Proposition 54 *Let $u:\Omega\subset\mathbb{R}^n\to\mathbb{R}\in C^{1,\beta}(B_{2\rho}(0))$ be a weak solution to $\Delta u=u^{-\alpha}$ such that $u\geq 0$ and $u(0)=0$.*

$$u\notin C^2(B_{2\rho}(0)) \text{ and } \beta\leq\frac{1-\alpha}{1+\alpha}$$

Proof. First, for any such function, $\nabla u(0)=0$. This comes from the requirement that $u\geq 0$. If $\nabla u(0)$ were not zero, then in some direction the linearization of u at zero in that direction would take on negative values in an ϵ-ball. Since we are assuming that $u\geq 0$, this would be a contradiction. Now, since $u\in C^{1,\beta}(B_{2\rho}(0))$, $\nabla u\in C^{0,\beta}(B_{2\rho}(0))$ and we have that for some $C\in\mathbb{R}$

$$|\nabla u(x)-\nabla u(0)|=|\nabla u(x)|\leq C|x-0|^{\beta}=C|x|^{\beta}$$

and we have that

$$\begin{aligned} u(x) &= \int_0^1\frac{d}{dt}(u(tx))\,dt=\int_0^1\nabla u(tx)\cdot x dt\leq\int_0^1|\nabla u(tx)||x|\,dt \\ &\leq C\int_0^1(t|x|)^{\beta}|x|\,dt=C|x|^{\beta+1}\frac{1}{\beta+1}\leq C|x|^{\beta+1} \end{aligned}$$

Construct a function $\zeta(x)$ as follows. Define a smooth function $\psi(r)$ so that $\psi(r) = 1$ for $0 \le r \le 1$ and $\psi(r) = 0$ for $r \ge 2$. For $1 \le r \le 2$ bound the first and second derivatives so that $\left|\psi'(r)\right| \le C_1$ and $\left|\psi''(r)\right| \le C_2$. Now set $\zeta(x) = \psi\left(\frac{|x|}{\rho}\right)$. This means $|\nabla\zeta| \le \frac{C_1}{\rho}$ and $|\Delta\zeta| \le \left|\frac{C_1}{\rho^2} + \frac{1}{n-1}\frac{C_2}{\rho^2}\right| = \left|\frac{C_3}{\rho^2}\right|$ where $C_3 = C_1 + \frac{C_2}{n-1}$. We now establish two inequalities for $\int u\Delta\zeta$:

$$\int\limits_{B_{2\rho}} u\Delta\zeta \le C\frac{C_2}{\rho^2}\int\limits_{B_{2\rho}} |x|^{\beta+1} = C\frac{C_3}{\rho^2}\omega_n \cdot n\int\limits_0^{2\rho} r^{\beta+1}r^{n-1}dr = \frac{nC \cdot C_2\omega_n}{(\beta+1+n)\rho^2}(2\rho)^{\beta+1+n}$$

$$\begin{aligned}\int\limits_{B_{2\rho}} u\Delta\zeta &= \int\limits_{B_{2\rho}} u^{-\alpha}\zeta \ge \int\limits_{B_\rho} u^{-\alpha}\zeta = \int\limits_{B_\rho} u^{-\alpha} \ge \int\limits_{B_\rho}\left(C|x|^{\beta+1}\right)^{-\alpha} = C^{-\alpha}\int\limits_{B_\rho}|x|^{-\alpha\beta-\alpha} \\ &= C^{-\alpha}\omega_n n\int\limits_0^\rho r^{-\alpha\beta-\alpha}r^{n-1}dr = \frac{C^{-\alpha}\omega_n n}{n-\alpha\beta-\alpha}\rho^{n-\alpha\beta-\alpha}\end{aligned}$$

Thus

$$\frac{C \cdot C_2 \cdot 2^{\beta+1+n}}{(\beta+1+n)}\rho^{\beta+n-1} \ge \frac{C^{-\alpha}}{n-\alpha\beta-\alpha}\rho^{n-\alpha\beta-\alpha}$$

Since ρ can be made very small, we need the power $\beta+n+1$ to be less than $n-\alpha\beta-\alpha$ or the inequalities could not possibly be satisfied. Thus

$$\begin{aligned}\beta+n-1 &\le n-\alpha\beta-\alpha \\ \beta+\alpha\beta &\le 1-\alpha \\ \beta &\le \frac{1-\alpha}{1+\alpha}\end{aligned}$$

and any weak solution of $\Delta u = u^{-\alpha}$ such that $u \ge 0$ and $u(0) = 0$ can be at most $C^{1,\frac{1-\alpha}{1+\alpha}}$ near zero.

■

Remark. $u = C_\alpha|x|^{2/(1+\alpha)}$ is a radially symmetric solution (not weak) to $\Delta u = u^{-\alpha}$, as long as we avoid any point where $u = 0$ and choose C_α correctly (the computation to derive C_α is elementary and C_α depends only on α and n). To show that it is a weak solution at the origin, we isolate a ball of radius ρ from Ω and perform some standard tricks to obtain

$$\begin{aligned}\int u\Delta\zeta &= \int\limits_{\Omega/B_\rho} u\Delta\zeta + \int\limits_{B_\rho} u\Delta\zeta \\ &= -\int\limits_{\partial B_\rho} \nabla u\cdot\nabla\zeta - \int\limits_{\partial B_\rho} \zeta\Delta u + \int\limits_{\Omega/B_\rho} u^{-\alpha}\zeta + \int\limits_{B_\rho} u\Delta\zeta\end{aligned}$$

The three integrands other than the $u^{-\alpha}\zeta$ term are all bounded functions, so as $\rho \to 0$, their integrals go to zero and the desired relation is obtained. Hence $u = C_\alpha |x|^{2/(1+\alpha)}$ is not a classical solution to $\Delta u = u^{-\alpha}$ for $0 < \alpha < 1$ near $u = 0$, but it is a classical solution away from $u = 0$.

The radially symmetric solution $u = C_\alpha |x|^{2/(1+\alpha)}$ is in fact $C^{1,\frac{1-\alpha}{1+\alpha}}$ near zero, so the bound on the Hölder coefficient we derived is a strong one:

$$u'(x) = \frac{du}{d|x|} = \frac{2C_\alpha}{1+\alpha}|x|^{\frac{2}{1+\alpha}-1} = \frac{2C_\alpha}{1+\alpha}|x|^{\frac{1-\alpha}{1+\alpha}}$$

and thus

$$|u'(x) - u'(y)| = \frac{2C_\alpha}{1+\alpha}\left||x|^{\frac{1-\alpha}{1+\alpha}} - |y|^{\frac{1-\alpha}{1+\alpha}}\right| \le \frac{2C_\alpha}{1+\alpha}|x-y|^{\frac{1-\alpha}{1+\alpha}}$$

Thus we have proven that no classical solution could possibly touch $u = 0$ and have, in the process, derived a sharp upper bound on the regularity of solutions which do touch $u = 0$.

This differential equation seems quite simple. Even so, the analysis required to establish existence and regularity of solutions was far from effortless. For differential equations arising in physical applications, existence and regularity are generally much more difficult to prove. That being said, the methodology described above has been successful and has, for the most part, been the only successful systematic approach to solving PDEs.

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