3-2005

# The norm of a composition operator with linear symbol acting on the Dirichlet space

Christopher Hammond
*Connecticut College*, cnham@conncoll.edu



## Recommended Citation

Hammond, Christopher, 2005: The norm of a composition operator with linear symbol acting on the Dirichlet space. *Journal of Mathematical Analysis and Applications*, **341**, *no. 2*, 636-639.

# The norm of a composition operator with linear symbol acting on the Dirichlet space

**Keywords**
Composition operator; Dirichlet space

**Comments**
Initially published in *Journal of Mathematical Analysis and Applications*, 303 (2005), no. 2, *499–508.*



DOI: 10.1016/j.jmaa.2004.08.049

http://dx.doi.org/10.1016/j.jmaa.2004.08.049

# The norm of a composition operator with linear symbol acting on the Dirichlet space

Christopher Hammond

*Connecticut College, Box 5384*

*270 Mohegan Avenue*

*New London, CT 06320, USA*

*E-mail address: cnham@conncoll.edu*

*Telephone: (860) 439-2647*

*Fax: (860) 439-2700*

**Abstract**

We obtain a representation for the norm of a composition operator on the Dirichlet space induced by a map of the form $\varphi(z) = az+b$. We compare this result to an upper bound for $\|C_\varphi\|$ that is valid whenever $\varphi$ is univalent. Our work relies heavily on an adjoint formula recently discovered by Gallardo-Gutiérrez and Montes-Rodríguez.

*Key words:* Composition operator, Dirichlet space

*Email address:* `cnham@conncoll.edu` (Christopher Hammond).

## 1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $dA$ denote normalized area measure on $\mathbb{D}$. The *Dirichlet space* $\mathcal{D}$ is the set of all analytic functions $f$ on $\mathbb{D}$ for which

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

is finite. The Dirichlet space is a Hilbert space under the obvious inner product. Moreover, $\mathcal{D}$ is a *reproducing kernel* Hilbert space; that is, for any point $w$ in $\mathbb{D}$, there is a corresponding $K_w$ in $\mathcal{D}$ such that $\langle f, K_w \rangle_{\mathcal{D}} = f(w)$ for all $f$ in $\mathcal{D}$. In the case of the Dirichlet space, these functions have the form $K_w(z) = 1 + \log(1/(1 - \overline{w}z))$. Note that

$$\|K_w\|_{\mathcal{D}}^2 = \langle K_w, K_w \rangle_{\mathcal{D}} = K_w(w) = 1 + \log\Big(1/\Big(1 - |w|^2\Big)\Big).$$

Given an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, we define the *composition operator* $C_\varphi$ on $\mathcal{D}$ by the rule

$$C_\varphi(f) = f \circ \varphi.$$

It is certainly not obvious that every $C_\varphi$ should take $\mathcal{D}$ into itself, and there are actually many examples for which it does not (see Proposition 3.12 in [9]). Nevertheless, any univalent $\varphi$ (or even any $\varphi$ with bounded valence) is guaranteed to induce a bounded composition operator on $\mathcal{D}$. This paper is only concerned with univalent $\varphi$, so the problem of unbounded composition operators will not arise. There are, of course, many other Hilbert spaces on which one can consider the action of a composition operator. We will make reference to several results pertaining to the Hardy space $H^2$ and the weighted Bergman spaces $A_\alpha^2$. Cowen and MacCluer's book [4] serves as a standard reference for such topics.

One of the major impediments to the study of (bounded) composition operators is the lack of a reasonable representation for the adjoint $C_\varphi^*$. It is well known, and is easy to prove, that $C_\varphi^*(K_w) = K_{\varphi(w)}$ for any reproducing kernel function $K_w$. Beyond this fact, not much is known about the adjoints of composition operators. Working on the Hardy space $H^2$, Cowen [3] obtained an explicit representation for $C_\varphi^*$ in the case where $\varphi$ is a linear fractional map; his result was later extended to the weighted Bergman spaces $A^2_\alpha$ by Hurst [8]. Their arguments rely heavily on the particular form of the reproducing kernel functions for those spaces, and hence cannot be adapted to the Dirichlet space. Gallardo-Gutiérrez and Montes-Rodríguez [5], however, have recently discovered a representation for $C_\varphi^* : \mathcal{D} \to \mathcal{D}$ when $\varphi$ is linear fractional. Their adjoint formula (which appears in Section 3 below) provides the foundation for the results of this paper.

One of Cowen's original applications for his adjoint formula was to determine the norm of an operator $C_\varphi : H^2 \to H^2$ when $\varphi$ has the form $\varphi(z) = az + b$. Similarly, Hurst was able to calculate the norms of the analogous operators on $A^2_\alpha$. (See also [12] and [13].) Now that we have an adjoint formula that is valid for the Dirichlet space, it seems reasonable to consider the same problem in this context. The actual result we obtain (Theorem 4.2) has a rather different appearance from its counterparts in the Hardy and weighted Bergman spaces.

The question of actually calculating the norm of a composition operator is not a trivial one. Aside from the aforementioned results of Cowen [3] and Hurst [8], there are not many instances for which we know the exact value of the norm. Even the case where $\varphi$ is linear fractional has proved quite difficult. Bourdon, Fry, Spofford, and the author [1] (and the author individually [6]) have considered this question in the context of the Hardy space. Gallardo-

Gutiérrez and Montes-Rodríguez [5], as a consequence of their work with adjoints, were able to determine the norm of such an operator acting on the subspace $\mathcal{D}_0 = \{f \in \mathcal{D} : f(0) = 0\}$. As we stated in the last paragraph, our goal here is much more modest: simply to obtain a representation for the norm of $C_\varphi : \mathcal{D} \to \mathcal{D}$ when $\varphi$ has the form $\varphi(z) = az + b$. As it turns out, the techniques we employ will be similar to those used to deal with the linear fractional case on the Hardy space.

In spite of the difficulties associated with computing the norm exactly, it is often possible to find sharp estimates for $\|C_\varphi\|$ in terms of the value $|\varphi(0)|$. Such results are well known for operators acting on the Hardy and weighted Bergman spaces (see Section 3.1 of [4]). Martín and Vukotić [10] recently obtained an upper bound for the norm of $C_\varphi : \mathcal{D} \to \mathcal{D}$ when $\varphi$ is univalent. We will discuss this result, and its relationship to our norm representation, in Section 5.

## 2 Preliminaries

Let $T$ be a bounded operator on a Hilbert space $\mathcal{H}$, with $T^*$ denoting its adjoint operator. Since the spectral radius of $T^*T$ equals $\|T^*T\| = \|T\|^2$, it seems reasonable to study the spectrum of $T^*T$ when trying to determine $\|T\|$. This point of view has served as the basis for much of the recent work relating to the norms of composition operators (e.g. [1] and [6]). Even the results of Cowen [3] and Hurst [8], when viewed in a particular manner, can be seen as statements pertaining to the spectrum of $C_\varphi^* C_\varphi$ (see Chapter 4 of [7]). The following proposition, which can be proved with a straightforward Hilbert space argument (see Proposition 1.2 in [7]), further emphasizes the connection between $\|T\|$ and the spectrum of $T^*T$.

**Proposition 2.1** *Let $h$ be an element of $\mathcal{H}$; then $\|T(h)\| = \|T\| \, \|h\|$ if and only if $(T^*T)(h) = \|T\|^2 \, h$.*

Whenever there is a nonzero $h$ such that $\|T(h)\| = \|T\| \, \|h\|$, we say that the operator $T$ is *norm-attaining*. Proposition 2.1 tells us that an operator $T$ has this property if and only if $\|T\|^2$ is an eigenvalue for $T^*T$. The next result, whose proof appears in [6], provides further insight into this situation. Recall that the *essential norm* $\|T\|_e$ of an operator $T : \mathcal{H} \to \mathcal{H}$ is simply the norm of its equivalence class in the Calkin algebra; that is,

$$\|T\|_e := \inf_K \|T - K\|,$$

the infimum being taken over all compact operators $K : \mathcal{H} \to \mathcal{H}$.

**Proposition 2.2** *If $\|T\|_e < \|T\|$, then the operator $T$ is norm-attaining.*

The object of this section is to show that the composition operators we are currently studying have the property that $\|C_\varphi\|_e < \|C_\varphi\|$, from which it will follow that they are norm-attaining. We begin with the following (widely known) result.

**Proposition 2.3** *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map that induces a bounded composition operator on $\mathcal{D}$; then $\|C_\varphi\| \geq \sqrt{1 + \log\Big(1/\Big(1 - |\varphi(0)|^2\Big)\Big)}$.*

**PROOF.** Observe that the kernel function $K_0(z) = 1$ is a unit vector and that

$$\left\|C_\varphi^*(K_0)\right\|_{\mathcal{D}}^2 = \left\|K_{\varphi(0)}\right\|_{\mathcal{D}}^2 = 1 + \log\Big(1/\Big(1 - |\varphi(0)|^2\Big)\Big).$$

Our claim follows immediately, since $\|C_\varphi\| = \|C_\varphi^*\|$. $\square$

Our next step is to obtain an estimate for the essential norm of $C_\varphi$.

**Proposition 2.4** *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a univalent map; then the essential norm of the operator $C_\varphi : \mathcal{D} \to \mathcal{D}$ is no greater than* 1.

**PROOF.** Let $P$ denote the orthogonal projection from $\mathcal{D}$ onto the subspace $\mathcal{D}_0$; that is, $(Pf)(z) = f(z) - f(0)$. Let $Q = I - P$, where $I$ denotes the identity map on $\mathcal{D}$. Since $Q$ is a compact operator, it follows that

$$\|C_\varphi\|_e = \|PC_\varphi + QC_\varphi\|_e = \|PC_\varphi\|_e \leq \|PC_\varphi\|.$$

It is not difficult to see that the operator $PC_\varphi$ is a contraction on $\mathcal{D}$. Because $\varphi$ is univalent, the standard change-of-variables formula simply reduces to $w = \varphi(z)$ and $dA(w) = |\varphi'(z)|^2 \, dA(z)$; therefore

$$\|(PC_\varphi)(f)\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(\varphi(z))|^2 \, |\varphi'(z)|^2 \, dA(z) = \int_{\varphi(\mathbb{D})} |f'(w)|^2 \, dA(w) \leq \|f\|_{\mathcal{D}}^2$$

for all $f$ in $\mathcal{D}$. Consequently $\|C_\varphi\|_e \leq 1$, as we had hoped to show. $\square$

Whenever $\varphi : \mathbb{D} \to \mathbb{D}$ is a univalent map with $\varphi(0) \neq 0$, Propositions 2.3 and 2.4 combine to show us that $\|C_\varphi\|_e < \|C_\varphi\|$. Therefore Proposition 2.2 dictates that such an operator $C_\varphi : \mathcal{D} \to \mathcal{D}$ is norm-attaining, a fact which will prove useful in Section 4.

## 3 Operators with linear fractional symbol

Our immediate goal is to obtain a functional equation that relates an eigenvalue of $C_\varphi^* C_\varphi$ to the values of its eigenfunctions at particular points in the disk. We will perform our preliminary work in the more general setting where $\varphi$ is a linear fractional map. To that end, let

$$\varphi(z) = \frac{az+b}{cz+d} \tag{1}$$

be a nonconstant linear fractional self-map of $\mathbb{D}$. As in the case of the Hardy and weighted Bergman spaces, the adjoint $C_\varphi^* : \mathcal{D} \to \mathcal{D}$ can be written in terms of

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}. \tag{2}$$

The map $\sigma$ takes $\mathbb{D}$ into itself whenever $\varphi$ has the same property (see Lemma 9.1 in [4]). Gallardo-Gutiérrez and Montes-Rodríguez [5] showed that the adjoint $C_\varphi^*$ on $\mathcal{D}$ can be written

$$\left(C_\varphi^* f\right)(z) = f(\sigma(z)) + f(0)K_{\varphi(0)}(z) - f(\sigma(0))$$

for any $f$ in $\mathcal{D}$. Following the convention established in [1] and [6], we write $\tau$ to denote the map

$$\tau(z) = (\varphi \circ \sigma)(z) = \frac{\left(|a|^2 - |b|^2\right)z + b\overline{d} - a\overline{c}}{\left(\overline{a}c - \overline{b}d\right)z + |d|^2 - |c|^2}. \tag{3}$$

(Gallardo-Gutiérrez and Montes-Rodríguez use $\phi$ to signify the same map.) It follows that the operator $C_\varphi^* C_\varphi : \mathcal{D} \to \mathcal{D}$ can be written

$$\left(C_\varphi^* C_\varphi f\right)(z) = f(\tau(z)) + f(\varphi(0))K_{\varphi(0)}(z) - f(\tau(0))$$

for all $z$ in $\mathbb{D}$, for any $f$ in $\mathcal{D}$. In particular, if $g$ is an eigenfunction for $C_\varphi^* C_\varphi$ corresponding to an eigenvalue $\lambda$, then

$$\lambda g(z) = g(\tau(z)) + g(\varphi(0))K_{\varphi(0)}(z) - g(\tau(0)). \tag{4}$$

We would like to exploit equation (4) to obtain information about the possible values of $\lambda$, and hence about the norm of $C_\varphi$. Our strategy is based largely on the methods developed in [1] and [6].

Throughout this paper, we write $\tau_j$ to denote the $j$th iterate of $\tau$; that is, $\tau_0$ is the identity map on $\mathbb{D}$ and $\tau_{j+1} = \tau \circ \tau_j$. Our next result is analogous to Proposition 5.1 in [6].

**Proposition 3.1** *Let $g$ be an eigenfunction for $C_\varphi^* C_\varphi$ corresponding to an eigenvalue $\lambda$. For any natural number $n$, the equation*

$$\lambda^n g(z) = g(\tau_n(z)) + \sum_{j=1}^{n} \lambda^{n-j} [g(\varphi(0)) K_{\varphi(0)}(\tau_{j-1}(z)) - g(\tau(0))]$$

*holds for all $z$ in $\mathbb{D}$.*

**PROOF.** This proposition follows from an elementary induction argument, the base case and the induction step both coming as consequences of equation (4). $\square$

Since $\|C_\varphi\|^2 > 1$ whenever $\varphi(0) \neq 0$, it seems fair to restrict our attention to the eigenvalues of $C_\varphi^* C_\varphi$ which are larger than 1. For such values of $\lambda$, we are able to obtain an "$n = \infty$" version of Proposition 3.1.

**Proposition 3.2** *Let $g$ be an eigenfunction for $C_\varphi^* C_\varphi$ corresponding to an eigenvalue $\lambda > 1$; then*

$$g(z) = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j [g(\varphi(0)) K_{\varphi(0)}(\tau_{j-1}(z)) - g(\tau(0))] \tag{5}$$

*for all $z$ in $\mathbb{D}$.*

**PROOF.** Our result will follow directly from Proposition 3.1, if only we can show that

$$\lim_{n\to\infty} \frac{g(\tau_n(z))}{\lambda^n} = 0$$

for any $z$ in $\mathbb{D}$. There are several cases to consider. First of all, if $\varphi$ happens to be an automorphism, then one can easily see that $\sigma = \varphi^{-1}$; hence $\tau$ is simply the identity map on $\mathbb{D}$, from which our claim follows. If the Denjoy–Wolff point $w_0$ of $\tau$ lies inside $\mathbb{D}$, then the terms $g(\tau_n(z))$ are converging pointwise to $g(w_0)$, in which case our claim also holds. Suppose then that $\varphi$ is not an automorphism and that $w_0$ lies on the unit circle $\partial\mathbb{D}$. In this case, the map $\tau$ must be of parabolic type; that is $\tau'(w_0) = 1$. This fact can be deduced directly from Lemma 5.1 in [2] (or rather the remark immediately following its proof), or as a consequence of Theorem 4.1 in [5]. Hence we can appeal to the argument used to prove Lemma 3.3 in [1] to see that, for every $z$ in $\mathbb{D}$, there is a constant $C$ such that $(1 - |\tau_n(z)|)^{-1} \leq Cn$. Thus it follows that

$$\begin{aligned} |g(\tau_n(z))| &\leq \|g\|_{\mathcal{D}} \left\|K_{\tau_n(z)}\right\|_{\mathcal{D}} \\ &= \|g\|_{\mathcal{D}} \sqrt{1 + \log\Big(1/\big(1 - |\tau_n(z)|^2\big)\Big)} \\ &\leq \|g\|_{\mathcal{D}} \sqrt{1 + \log(Cn)}, \end{aligned}$$

from which we obtain the desired result. $\square$

Unfortunately, Proposition 3.2 (which is based on the proof of Theorem 3.5 in [1]) does not allow us readily to determine $\|C_\varphi\|$. The fact that expression (5) involves both $g(\varphi(0))$ and $g(\tau(0))$ prevents us from obtaining an equation solely in terms of $\lambda$. The maps in which we are most interested, though, are precisely those for which $\varphi(0) = \tau(0)$. In that case, as we shall see in the next section, Proposition 3.2 will allow us to find a series representation for $\|C_\varphi\|^2$.

## 4 Operators with linear symbol

For the remainder of our discussion, we will restrict our attention to the setting where $\varphi$ has the form $\varphi(z) = az+b$, with $a$ and $b$ both nonzero and $|a|+|b| \le 1$. (Note that $b = 0$ implies $\|C_\varphi\| = 1$.) We can, of course, view $\varphi$ as being a linear fractional map, as in (1), with $c = 0$ and $d = 1$. Define the maps $\sigma$ and $\tau$ as in (2) and (3). In this case, we see that $\sigma(0) = 0$, which means that $\tau(0) = \varphi(0)$. Hence Proposition 3.2 becomes a more manageable result; equation (5) can be rewritten

$$\begin{aligned} g(z) &= g(\tau(0)) \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j [K_{\tau(0)}(\tau_{j-1}(z)) - 1] \\ &= g(\tau(0)) \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j \log\Big(1/\big(1 - \bar{b}\tau_{j-1}(z)\big)\Big). \end{aligned} \tag{6}$$

If $g(\tau(0)) = 0$, then equation (6) would dictate that the function $g(z)$ is identically 0. Thus any eigenfunction $g$ must have the property that $g(\tau(0)) \neq 0$. Therefore, taking $z = \tau(0)$, we see that any eigenvalue $\lambda$ of $C_\varphi^* C_\varphi$ (with $\lambda > 1$) must satisfy the condition

$$1 = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j \log\left(\frac{1}{1 - \bar{b}\tau_j(0)}\right). \tag{7}$$

Our main result will be stated in terms of this equation.

At this point, we will briefly discuss the possible solutions to equation (7). The following lemma will greatly simplify the situation.

**Lemma 4.1** *For any $j \ge 1$, the point $\bar{b}\tau_j(0)$ is a positive real number; moreover, $\bar{b}\tau_j(0) < \bar{b}w_0$, where $w_0$ denotes the Denjoy–Wolff point of $\tau$.*

**PROOF.** In the case we are considering, the map $\tau$ has the form

$$\tau(z) = \frac{\left(|a|^2 - |b|^2\right)z + b}{-\bar{b}z + 1}.$$

A simple calculation shows that

$$w_0 = \frac{1 - |a|^2 + |b|^2 - \sqrt{\left(1 - |a|^2 + |b|^2\right)^2 - 4\,|b|^2}}{2\bar{b}}.$$

The numerator of this expression is a real number greater than $2\,|b|^2$, which means that $\bar{b}w_0$ is a positive real number, and also that $|w_0| > |b|$. Thus our claim is equivalent to saying that every point $\tau_j(0)$ belongs to the set $\mathcal{S} = \left\{tb : 0 < t < \bar{b}w_0/\,|b|^2\right\}$, the line segment connecting the points 0 and $w_0$. Consider the image of $\mathcal{S}$ under the linear fractional map $\tau$. The image must be an open-ended line segment inside $\mathbb{D}$. (The image cannot be an arc, since the point $\tau^{-1}(\infty) = b/\,|b|^2$ lies on the line containing $\mathcal{S}$.) Since $\tau(w_0) = w_0$, one endpoint of the image segment must be $w_0$; the other endpoint is $\tau(0) = b$. Since $b$ itself belongs to $\mathcal{S}$, we conclude that the image of $\mathcal{S}$ under $\tau$ must actually be a subsegment of $\mathcal{S}$. Thus, by induction, we see that each point $\tau_j(0)$ does indeed lie on the segment $\mathcal{S}$. $\square$

Consider the analytic function

$$F(z) = \sum_{j=1}^{\infty} \log\left(\frac{1}{1 - \bar{b}\tau_j(0)}\right) z^j. \tag{8}$$

Lemma 4.1 guarantees that each of the coefficients $\log\left(1/\left(1 - \bar{b}\tau_j(0)\right)\right)$ is a positive real number. Therefore, since the points $\tau_j(0)$ converge to $w_0$, we make the following elementary observations:

- The power series that defines $F(z)$ has radius of convergence 1.
- $F(x)$ is a non-negative real number for all $x$ in the interval $[0, 1)$.

- $F(0) = 0$ and the series $F(1)$ diverges to infinity.
- $F'(x) > 0$ for all $x$ in the interval $(0, 1)$.

Based on this information, we conclude that there is exactly one $\xi$ in the interval $(0, 1)$ such that $F(\xi) = 1$. In other words, there is exactly one number $\lambda > 1$ that satisfies equation (7).

Combining the results of the last three sections, we obtain the following characterization of $\|C_\varphi\|$.

**Theorem 4.2** *Let $\varphi(z) = az+b$, where $a$ and $b$ are both nonzero and $|a|+|b| \leq 1$, and consider the operator $C_\varphi : \mathcal{D} \to \mathcal{D}$. Then $\lambda = \|C_\varphi\|^2$ is the unique positive real solution to the equation*

$$1 = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j \log\left(\frac{1}{1 - \bar{b}\tau_j(0)}\right),$$

*where $\tau_j$ denotes the $j$th iterate of the map*

$$\tau(z) = \frac{\left(|a|^2 - |b|^2\right)z + b}{-\bar{b}z + 1}.$$

**PROOF.** The arguments set forth in Section 2 show that $\|C_\varphi\|^2$ is an eigenvalue for $C_\varphi^* C_\varphi$, with $\|C_\varphi\|^2 > 1$. Hence $\lambda = \|C_\varphi\|^2$ is a positive real solution to equation (7). We have just observed that only one such solution exists. □

*Remark.* In certain instances, our norm representation takes on a somewhat more tractable form. In particular, consider the maps $\varphi(z) = az + b$ with $0 < |b| < 1$ and $|a| + |b| = 1$. For such $\varphi$, an induction argument shows that

$$\tau_j(z) = \frac{(1 - (j+1)\,|b|)\,z + jb}{-j\bar{b}z + 1 + (j-1)\,|b|}$$

for all $j \geq 1$. Hence equation (7) can be rewritten

$$1 = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j \log\left(\frac{1 + (j-1)\,|b|}{(1 - |b|)\,(1 + j\,|b|)}\right). \tag{9}$$

For example, take $\varphi(z) = (1/2)\,z + 1/2$; in this case, equation (9) simply becomes

$$1 = \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j \log\left(\frac{2j+2}{j+2}\right).$$

Using numerical methods, we see that $\|C_\varphi\| = \sqrt{\lambda} \approx 1.195830076$.

## 5 Operators with maximal norm

As we mentioned in the introduction, Martín and Vukotić [10] recently obtained an upper bound for the norm of $C_\varphi : \mathcal{D} \to \mathcal{D}$ when $\varphi$ is univalent; in particular,

$$\|C_\varphi\| \leq \sqrt{\frac{2 + L + \sqrt{L(4+L)}}{2}}, \tag{10}$$

where $L = \log\Big(1/\Big(1 - |\varphi(0)|^2\Big)\Big)$. Moreover, they showed that equality occurs whenever $\varphi$ is a *full map*, that is, the area of $\mathbb{D} \setminus \varphi(\mathbb{D})$ equals 0. Whenever there is equality in (10) for some univalent $\varphi$, we say that the operator $C_\varphi$ has *maximal norm* on $\mathcal{D}$. In light of the work done by Joel Shapiro [11] on the Hardy space, it seems reasonable to try to determine which composition operators possess this property. We have already noted that any univalent full map induces a composition operator with maximal norm. It also follows from the work of Martín and Vukotić (in particular, Theorem 2 in [10]) that, whenever $\varphi(0) \neq 0$, the operator $C_\varphi$ cannot have maximal norm unless $\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in \mathbb{D}\} = 1$. We can actually use our own norm representation to contribute a small piece of information to this line of inquiry: namely that no map of the form $\varphi(z) = az + b$, with $b \neq 0$, induces a composition operator with maximal norm on $\mathcal{D}$. The substance of this assertion lies in the following estimate. (The reader will notice a certain symmetry between this result and Proposition 2.3.)

**Proposition 5.1** *Suppose that $\varphi(z) = az + b$, where $a$ and $b$ are both nonzero and $|a| + |b| \leq 1$; then*

$$\|C_\varphi\| \leq \sqrt{1 + \log\left(\frac{1}{1 - \bar{b}w_0}\right)}, \tag{11}$$

*where $w_0$ denotes the Denjoy–Wolff point of $\tau$.*

**PROOF.** Consider the analytic function $F(z)$, as defined in line (8). Bearing in mind the result of Lemma 4.1, we see that

$$\log\left(\frac{1}{1 - \bar{b}\tau_j(0)}\right) \leq \log\left(\frac{1}{1 - \bar{b}w_0}\right)$$

for all $j \geq 1$. Hence, for any $x$ in the interval $[0, 1)$, it follows that

$$F(x) \leq \sum_{j=1}^{\infty} \log\left(\frac{1}{1 - \bar{b}w_0}\right) x^j = \frac{x}{1 - x} \log\left(\frac{1}{1 - \bar{b}w_0}\right). \tag{12}$$

We have already noted that there is a unique $\xi$ in $(0, 1)$ such that $F(\xi) = 1$; furthermore, we know that $F'(x) > 0$ on $(0, 1)$. Hence (12) shows that $\xi$ is greater than or equal to any $x$ satisfying the equation

$$\frac{x}{1 - x} \log\left(\frac{1}{1 - \bar{b}w_0}\right) = 1.$$

In other words,

$$\xi \geq \left(1 + \log\left(\frac{1}{1 - \bar{b}w_0}\right)\right)^{-1}.$$

Our claim follows from Theorem 4.2, which dictates that $\|C_\varphi\| = \sqrt{1/\xi}$. $\square$

In the case where $|a| + |b| < 1$, Proposition 5.1 has an interesting geometrical interpretation. Appealing to Lemma 4 in [12], we see that line (11) can be rewritten

$$\|C_\varphi\| \le \sqrt{1+\log\left(\frac{r}{R}\right)},$$

where $R$ denotes the Euclidean radius of the disk $\varphi(\mathbb{D})$ and $r$ its pseudohyperbolic radius. When $|a|+|b|=1$, on the other hand, our estimate simply becomes

$$\|C_\varphi\| \le \sqrt{1+\log\left(\frac{1}{1-|b|}\right)}.$$

This last statement will allow us to obtain our final result.

**Theorem 5.2** *Let $\varphi(z)=az+b$, where $a$ and $b$ are both nonzero and $|a|+|b|\le 1$. The operator $C_\varphi:\mathcal{D}\to\mathcal{D}$ does not have maximal norm.*

**PROOF.** If $|a|+|b|<1$, then $\|\varphi\|_\infty<1$. As we have already mentioned, Theorem 2 in [10] guarantees that no such map induces a composition operator with maximal norm. Suppose then that $|a|+|b|=1$. In view of Proposition 5.1, we simply need to show that, for any $b=\varphi(0)\ne 0$, the quantity $\sqrt{1+\log(1/(1-|b|))}$ is strictly less than the term on right-hand side of (10). After some manipulation, this claim reduces to the easily verifiable inequality

$$\log(1-|b|)+\log(1+|b|)<\log(1-|b|)\log(1+|b|).$$

Therefore the upper bound stated in Proposition 5.1 is strictly less than the bound obtained by Martín and Vukotić [10]. In other words, none of the maps we are considering induces a composition operator with maximal norm. $\square$

## References

[1] P. S. Bourdon, E. E. Fry, C. Hammond, C. H. Spofford, Norms of linear-fractional composition operators, Trans. Amer. Math. Soc. 356 (2004) 2459–2480.

[2] P. S. Bourdon, D. Levi, S. K. Narayan, J. H. Shapiro, Which linear-fractional composition operators are essentially normal?, J. Math. Anal. Appl. 280 (2003) 30–53.

[3] C. C. Cowen, Linear fractional composition operators on $H^2$, Integral Equations Operator Theory 11 (1988) 151–160.

[4] C. C. Cowen, B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.

[5] E. A. Gallardo-Gutiérrez, A. Montes-Rodríguez, Adjoints of linear fractional composition operators on the Dirichlet space, Math. Ann. 327 (2003) 117–134.

[6] C. Hammond, On the norm of a composition operator with linear fractional symbol, Acta Sci. Math. (Szeged) 69 (2003) 813–829.

[7] C. Hammond, On the norm of a composition operator, Ph.D. thesis, University of Virginia, 2003.

[8] P. R. Hurst, Relating composition operators on different weighted Hardy spaces, Arch. Math. (Basel) 68 (1997) 503–513.

[9] B. D. MacCluer, J. H. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canad. J. Math. 38 (1986) 878–906.

[10] M. J. Martín, D. Vukotić, Norms and spectral radii of composition operators acting on the Dirichlet space, J. Math. Anal. Appl., to appear.

[11] J. H. Shapiro, What do composition operators know about inner functions?, Monatsh. Math. 130 (2000) 57–70.

[12] D. Vukotić, On norms of composition operators acting on Bergman spaces, J. Math. Anal. Appl. 291 (2004) 189–202.

[13] D. Vukotić, Erratum to "On norms of composition operators acting on Bergman spaces" [J. Math. Anal. Appl. 291 (2004) 189–202], preprint, 2004.