Numerical Black Hole Solutions For A Timelike Bumblebee Field

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Abstract

This work builds on a paper by Casana et al. [1], in which they calculated an exact, spherically symmetric solution to the Einstein equations in the presence of a bumblebee vector field. We extend their solutions by introducing a timelike component to the vector field. We are able to obtain series solutions to the Einstein equation in the presence of this field. We then demonstrate that these solutions recover flat spacetime. Our results cast doubt on the physical viability of this model, but a more thorough proof is required.

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Introduction

1.1 Background

Lorentz symmetry explains that the laws of physics for all observers are the same within an inertial reference frame. The principle was first introduced by Albert Einstein in his 1905 paper on special relativity [2]. It has since survived over one hundred years of tests to form the foundation of much modern physics, including general relativity and the Standard Model. The Standard Model successfully describes phenomena at quantum scales but falls short for large scales. On the other hand, general relativity is effective on large scales but not on small scales. Therefore, unifying these two theories would be of great significance.

Many proposed methods of unification, including string theory and loop quantum gravity, lead to predictions of Lorentz symmetry violation [3][4][5]. Because Lorentz symmetry underlies much of the mathematics of general relativity and the Standard Model, its violation presents an opportunity to investigate the shortcomings of these theories.

In 1997, Colladay and Kostelecký introduced the Standard Model Extension (SME), a framework covering all possible Lorentz and CPT-violating coefficients [6][7]. The SME also integrates special relativity, general relativity, and the Standard Model, providing information about possible signals of Lorentz violation and opening up new experimental searches. It adds terms to the Standard Model that account for Lorentz violations and become significant at the Planck scale [8]. The SME has already been extensively studied, including the electromagnetic sector [9][10][11][12][13], the electroweak sector [14], the strong sector [15], and in hadronic physics [16]. Some of the gravitational sector has also been studied, especially the effects of Lorentz violation on gravitational waves [17][18][19]. As of the writing of this thesis, no observations of Lorentz violation have been made, and the bounds on the coefficients predicted by the SME are continually tightening [20].

Most of the work done which has looked at gravity using the SME focuses on weak gravity models. This work aims to consider the SME as it relates to strong gravity, such as black holes. We focus on bumblebee gravity, a model of strong gravity that relies on a fundamental vector field that allows for a spontaneous violation of Lorentz symmetry.

1.2 Overview

This work extends the results obtained by Casana et al. in ref. [1], in which they calculated an exact, spherically symmetric solution to the Einstein equations in the presence of the bumblebee field. This field breaks Lorentz symmetry by assuming a nonzero vacuum expectation value. We modify their field by introducing a timelike component. Using code written in Mathematica with the package xTensor[21], we obtain solutions to the Einstein equation and demonstrate that this technique recovers flat spacetime[1]. Section 2 presents the theoretical and mathematical framework necessary to calculate

[1] A copy of the code written for this project will be available on the Connecticut College Digital Commons (digitalcommons.conncoll.edu/)

the Einstein equation generated by bumblebee gravity and the nonvanishing components of our Einstein tensor. In Section 3, we present our solutions to the Einstein equation. In Section 4, we discuss the implications of our results. In Section 5, we conclude that our solutions cast doubt on the viability of this model and suggest follow-up studies. Appendix A.1 presents a technical overview of the code we developed to aid our calculations. Appendices A.2 and A.3 present the full form of our series solutions and their simplification.

Theory

2.1 Standard Model Extension

The Standard Model Extension (SME) for flat spacetime was developed by Don Colladay and Alan Kostelecký [6][7] and was later expanded to include Lorentz-breaking terms in curved spacetime [22]. The field theory contains the Standard Model, general relativity, and all possible operators that break Lorentz symmetry.

The SME provides a framework that allows for the study of Lorentz violation (and CPT violation, which implies the breaking of Lorentz symmetry) by introducing terms that describe these possible violations.

It has been shown that of the two forms of Lorentz symmetry breaking, explicit and spontaneous breaking, only spontaneous breaking is compatible with laws governing the conservation of energy, momentum, and spin density [22]. Spontaneous symmetry breaking describes a situation in which a symmetric system spontaneously achieves an asymmetric state. Many models rely on a field with a nonzero vacuum expectation value (VEV) to achieve spontaneous symmetry breaking. This involves minimizing the field's potential energy when the field is nonzero. Such a potential causes the VEV to be nonzero in a vacuum, thereby allowing a system whose Lagrangian obeys symmetry to break that symmetry when the system is in a vacuum solution. The Higgs field is a well-known example of such a field in that it is a scalar field that breaks gauge symmetry by having a nonzero VEV [23]. Similarly, each term in the SME could arise from the expectation value of a tensor field that spontaneously breaks Lorentz symmetry.

2.2 Bumblebee Gravity

Bumblebee gravity is a model in which an underlying bumblebee vector field, B_μ, violates Lorentz symmetry by assuming a nonzero vacuum expectation value [22]. It is the simplest of the models that consider Lorentz violation in strongly curved spacetime and allows an avenue to study Lorentz violation within strong gravity models.

The action for such a field, as coupled to gravity and matter and assuming no torsion or cosmological constant, can be expressed as

$$S_B = \int \zeta \left(\frac{1}{2\kappa} R + \frac{1}{2\kappa} \xi B^\mu B^\nu R_{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - V(B^\mu) + \mathcal{L}_M \right) \mathrm{d}^4 x, \tag{2.2.1}$$

where $\zeta \equiv \sqrt{-g}$, $\kappa = 8\pi G_N$, ξ is the real coupling constant, and $\mathcal{L}_M$ is the Lagrangian for the conventional matter field. The field strength is defined as

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{2.2.2}$$

The potential V is defined to be a function of the quantity $B^\mu B_\mu \pm b^2$, i.e.

$$V \equiv V(B^\mu B_\mu \pm b^2), \tag{2.2.3}$$

so that B_μ has a nonzero VEV. If the field is in the vacuum manifold, the set of all points at which the potential is minimized, we expect that $V = 0$ and $V' = 0$. Therefore, Equation 2.2.3 can be simplified to

$$B^\mu B_\mu \pm b^2 = 0 \tag{2.2.4}$$

where b^2 is a positive real constant. Therefore, B^μ assumes a nonzero expectation value; call it

$$\left\langle B^\mu \right\rangle = b^\mu, \tag{2.2.5}$$

such that $b^\mu b_\mu = \pm b^2$. While the potential here is Lorentz invariant, the VEV breaks Lorentz symmetry, meaning there is a spontaneous breaking of Lorentz symmetry at those solutions.

The stress-energy tensor associated with this field has the form

$$T_{\mu\nu} = T^M_{\mu\nu} + T^B_{\mu\nu} \tag{2.2.6}$$

where $T^M_{\mu\nu}$ is the stress-energy from the matter sector, and $T^B_{\mu\nu}$ is that of the bumblebee field. Imposing $T^M_{\mu\nu} = 0$, we can limit our solutions to vacuum solutions–those describing the empty space surrounding a gravitating body. Therefore,

$$\begin{aligned} T_{\mu\nu} = T^B_{\mu\nu} = &- B_{\mu\alpha}B^\alpha_\nu - \frac{1}{4}B_{\mu\alpha}B^{\alpha\beta}g_{\mu\nu} - Vg_{\mu\nu} + 2V'B_\mu B_\nu \\ &+ \frac{\xi}{\kappa}\Bigg[\frac{1}{2}B^\alpha B^\beta R_{\alpha\beta}g_{\mu\nu} - B_\mu B^\alpha R_{\alpha\nu} - B_\nu B^\alpha R_{\alpha\mu} + \frac{1}{2}\nabla_\alpha\nabla_\mu(B^\alpha B^\nu) + \frac{1}{2}\nabla_\alpha\nabla_\nu(B^\alpha B^\mu) \\ &- \frac{1}{2}\nabla^2(B_\mu B_\nu) - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta(B^\alpha B^\beta)\Bigg]. \end{aligned} \tag{2.2.7}$$

The Einstein equation, which is the equation of motion associated with the metric $G_{\mu\nu}$, can therefore be written as

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.2.8}$$

where $G_{\mu\nu}$ is the Einstein tensor. Similarly, the action (2.2.1) can be varied with respect to the vector field B_μ to obtain the equation of motion for the field as

$$\nabla^\mu B_{\mu\nu} = J_\nu = J^B_\nu + J^M_\nu, \tag{2.2.9}$$

where J^M_ν is the contribution to the equations of motion from the matter sector, which vanishes in the vacuum solutions, and J^B_ν is from the bumblebee self-interaction and is defined as

$$J^B_\nu = 2V'B_\nu - \frac{\xi}{\kappa}B^\mu R_{\mu\nu}. \tag{2.2.10}$$

2.3 Metric

Casana et al. [1] utilized a Birkhoff metric in their research to obtain static and spherically symmetric vacuum solutions for the extended Einstein equations. A Birkhoff metric is defined as

$$g_{\mu\nu} = \mathrm{diag}(-e^{2\gamma}, e^{2\rho}, r^2, r^2\sin^2\theta) \tag{2.3.1}$$

where γ and ρ are both functions of r. In their paper, Casana and their colleagues assumed a spacelike background field of the form

$$b_\mu = (0, b_r, 0, 0). \tag{2.3.2}$$

The primary difference between their work and ours is that we introduce a timelike component to the background field. This change is motivated primarily by the fact that a purely radial vector field would always point in the direction of the black hole at its origin. Therefore, it would imply the existence of some black hole which, as the origin of the vector field, would constitute the "center of the universe." This goes against the cosmological principle, which states that the universe is homogeneous and isotropic on large scales. Adding a time component to the vector field allows it to be timelike, meaning it would behave like flat spacetime asymptotically, thereby preserving homogeneity and isotropy.

Implementing this addition gives the background field the form

$$b_\mu = (b_t, b_r, 0, 0). \tag{2.3.3}$$

The derivation for the exact form of the time component of this equation is as follows. Given $B^\mu B_\mu = \pm b^2$, decomposing the norm into components gives

$$g^{tt} b_t^2 + g^{rr} b_r^2 = \pm b^2. \tag{2.3.4}$$

It follows from the identity

$$g^{\mu\nu} g_{\mu\beta} = \delta^\mu_\beta \tag{2.3.5}$$

that the raised form of the Birkhoff metric given in 2.3.1 is

$$g^{\mu\nu} = \mathrm{diag}(-e^{-2\gamma}, e^{-2\rho}, r^{-2}, r^{-2}\csc^2\theta). \tag{2.3.6}$$

Substituting the relevant components into 2.3.4 gives

$$-e^{-2\gamma} b_t^2 + e^{-2\rho} b_r^2 = \pm b^2, \tag{2.3.7}$$

which implies

$$b_t = \left[e^{2\gamma} (e^{-2\rho} b_r^2 \pm b^2) \right]^{1/2}. \tag{2.3.8}$$

We also rearrange this result for b_r in terms of b_t to find

$$b_r = \left[e^{2\rho} (\pm b^2 e^{-2\gamma} + b_t^2) \right]^{1/2}. \tag{2.3.9}$$

2.4 Solving for the Einstein Tensor and Equations of Motion

Having derived the components of our field, the next step is to find the field equations. By noting that $V = 0$, as shown in the simplification to Equation 2.2.3, and that $B_{\mu\nu}$ takes a restricted form, we can simplify Equation 2.2.7 to

$$\begin{aligned} T^B_{\mu\nu} = &- B_{\mu\alpha} B^\alpha_\nu - \frac{1}{4} B_{\mu\alpha} B^{\alpha\beta} g_{\mu\nu} \\ &+ \frac{\xi}{\kappa} \Bigg[\frac{1}{2} B^\alpha B^\beta R_{\alpha\beta} g_{\mu\nu} - B_\mu B^\alpha R_{\alpha\nu} - B_\nu B^\alpha R_{\alpha\mu} + \frac{1}{2} \nabla_\alpha \nabla_\mu (B^\alpha B^\nu) + \frac{1}{2} \nabla_\alpha \nabla_\nu (B^\alpha B^\mu) \\ &- \frac{1}{2} \nabla^2 (B_\mu B_\nu) - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \nabla_\beta (B^\alpha B^\beta) \Bigg]. \end{aligned} \tag{2.4.1}$$

We rearrange the Einstein equation,2.2.8, such that

$$G_{\mu\nu} - \kappa T_{\mu\nu} = 0. \tag{2.4.2}$$

Substituting in the simplified stress-energy tensor 2.4.1, setting $\pm b^2 = -1$, and setting our constants to unity allows us to find the nonvanishing components of the Einstein tensor.

The components found using the form of the vector field in terms of b_r (2.3.8) are

$$G_{tt}-\kappa T_{tt} = \frac{1}{2r^2\left(b_r^2+e^{2\rho}\right)}\Bigg(e^{-2(\gamma+2\rho)}\Bigg(-\Bigg(b_r^2\Bigg(2rb_rb_r'\left(r\gamma'-r\rho'+2\frac{\xi}{\kappa}\right)$$
$$+r^2b_r'^2+b_r^2\Bigg(r\Bigg(2r\frac{\xi}{\kappa}\gamma''+2\gamma'\left(2\frac{\xi}{\kappa}-r(\frac{\xi}{\kappa}+1)\rho'\right)$$
$$+(2r\frac{\xi}{\kappa}+r)\gamma'^2-8\frac{\xi}{\kappa}\rho'+r\rho'^2\Bigg)+2\frac{\xi}{\kappa}\Bigg)\Bigg)\Bigg)$$
$$-2b_re^{2\rho}\Bigg(rb_r'\left(r\gamma'+2\frac{\xi}{\kappa}\right)+b_r\Bigg(r\Bigg(2r\frac{\xi}{\kappa}\gamma''+\left(\gamma'-\rho'\right)\left((2r\frac{\xi}{\kappa}+r)\gamma'+4\frac{\xi}{\kappa}\right)-2\rho'\Bigg)$$
$$+\frac{\xi}{\kappa}+1\Bigg)\Bigg)$$
$$+e^{4\rho}\Bigg(2b_r^2-r\Bigg(2r\frac{\xi}{\kappa}\gamma''+\gamma'\left((2r\frac{\xi}{\kappa}+r)\gamma'-2r\frac{\xi}{\kappa}\rho'+4\frac{\xi}{\kappa}\right)-4\rho'\Bigg)-2\Bigg)+2e^{6\rho}\Bigg)$$
$$2r^2\left(b_r^2+e^{2\rho}\right)\Bigg)\Bigg) \tag{2.4.3}$$

$$G_{rr}-\kappa T_{rr} = \frac{1}{2r^2\left(b_r^2+e^{2\rho}\right)}\Bigg(e^{-6\rho}\Bigg(b_r^2\Bigg(2rb_rb_r'\left(r\gamma'-r\rho'+2\frac{\xi}{\kappa}\right)$$
$$+r^2b_r'^2+b_r^2\Bigg(r\Bigg(r\left(\rho'^2-2\frac{\xi}{\kappa}\gamma''\right)+2\gamma'\left(r(\frac{\xi}{\kappa}-1)\rho'+2\frac{\xi}{\kappa}\right)+(r-2r\frac{\xi}{\kappa})\gamma'^2\Bigg)+2\frac{\xi}{\kappa}\Bigg)\Bigg)$$
$$-2b_re^{2\rho}\Bigg(b_r\Bigg(r\Bigg(r\frac{\xi}{\kappa}\gamma''+\gamma'\left(r(\frac{\xi}{\kappa}-1)\left(\gamma'-\rho'\right)-2(\frac{\xi}{\kappa}+1)\right)\Bigg)-\frac{\xi}{\kappa}-1\Bigg)$$
$$-rb_r'\left(r\gamma'+2\frac{\xi}{\kappa}\right)\Bigg)+e^{4\rho}\left(-2b_r^2+r\gamma'\left(r\gamma'+4\right)+2\right)-2e^{6\rho}\Bigg)\Bigg) \tag{2.4.4}$$

$$G_{\theta\theta}-\kappa T_{\theta\theta} = \sin^2\theta G_{\phi\phi}-\kappa T_{\phi\phi} = \frac{1}{2r^3\left(b_r^2+e^{2\rho}\right)}$$
$$\Bigg(e^{-4\rho}\Bigg(b_r^2\Bigg(r(2\frac{\xi}{\kappa}-1)b_r'^2+2b_r\Bigg(r\frac{\xi}{\kappa}b_r''+b_r'\left(r(3\frac{\xi}{\kappa}-1)\gamma'+r(1-5\frac{\xi}{\kappa})\rho'+2\frac{\xi}{\kappa}\right)\Bigg)\Bigg)$$
$$+b_r^2\Bigg(2r\frac{\xi}{\kappa}\left(\gamma''-\rho''\right)+2\gamma'\left((r-4r\frac{\xi}{\kappa})\rho'+\frac{\xi}{\kappa}\right)+r(2\frac{\xi}{\kappa}-1)\gamma'^2+r(6\frac{\xi}{\kappa}-1)\rho'^2-6\frac{\xi}{\kappa}\rho'\Bigg)\Bigg)$$
$$+2e^{2\rho}\Bigg(r\frac{\xi}{\kappa}b_r'^2+b_r\Bigg(r\frac{\xi}{\kappa}b_r''+b_r'\left(r(3\frac{\xi}{\kappa}-1)\gamma'+\frac{\xi}{\kappa}\left(2-5r\rho'\right)\right)\Bigg)\Bigg)$$
$$+b_r^2\Bigg(r(\frac{\xi}{\kappa}+1)\gamma''+\gamma'\left(-4r\frac{\xi}{\kappa}\rho'+\frac{\xi}{\kappa}+1\right)+r\frac{\xi}{\kappa}\gamma'^2-r\frac{\xi}{\kappa}\rho''+3r\frac{\xi}{\kappa}\rho'^2-(3\frac{\xi}{\kappa}+1)\rho'\Bigg)\Bigg)$$
$$+e^{4\rho}\left(2r\gamma''+\gamma'\left(r\gamma'-2r\rho'+2\right)-2\rho'\right)\Bigg)\Bigg) \tag{2.4.5}$$

where primes indicate derivatives with respect to r.

The same process using the form of the vector field in terms of b_t 2.3.9 results in the components

$$G_{tt}-\kappa T_{tt} = \frac{1}{2r^2}e^{-2(2\gamma+\rho)}\Bigg(-4r\frac{\xi}{\kappa}b_tb_t'-r^2b_t'^2-2\frac{\xi}{\kappa}b_t^2\left(r\left(r\left(\gamma''-\gamma'\rho'+\gamma'^2\right)-2\rho'\right)+1\right)$$
$$+2e^{2\gamma}\left(-2r(\frac{\xi}{\kappa}-1)\rho'+e^{2\rho}+\frac{\xi}{\kappa}-1\right)\Bigg) \tag{2.4.6}$$

$$G_{rr}-\kappa T_{rr}=\frac{1}{2r^2}e^{-2(\gamma+2\rho)}\Bigg(4r\frac{\xi}{\kappa}b_tb_t'+r^2b_t'^2+2\frac{\xi}{\kappa}b_t^2\Bigg(r\Bigg(2\rho'-r\Big(\gamma''-\gamma'\rho'+\gamma'^2\Big)\Bigg)+1\Bigg)$$
$$-2e^{2\gamma}\Bigg(r\Bigg(\frac{\xi}{\kappa}\Big(2\rho'-r\gamma''\Big)+\gamma'\Big(r\frac{\xi}{\kappa}\rho'+2\frac{\xi}{\kappa}-2\Big)+r(-\frac{\xi}{\kappa})\gamma'^2\Bigg)+e^{2\rho}+\frac{\xi}{\kappa}-1\Bigg)\Bigg) \tag{2.4.7}$$

$$G_{\theta\theta}-\kappa T_{\theta\theta}=\sin^2\theta G_{\phi\phi}-\kappa T_{\phi\phi}=\frac{1}{2r^3}e^{-2(\gamma+\rho)}\Bigg(r(2\frac{\xi}{\kappa}-1)b_t'^2+2\frac{\xi}{\kappa}b_t\Bigg(rb_t''-b_t'\Big(r\Big(\gamma'+\rho'\Big)-2\Big)\Bigg)$$
$$-2\frac{\xi}{\kappa}b_t^2\Big(\gamma'+\rho'\Big)-2(\frac{\xi}{\kappa}-1)e^{2\gamma}\Bigg(r\gamma''+\Big(r\gamma'+1\Big)\Big(\gamma'-\rho'\Big)\Bigg)\Bigg). \tag{2.4.8}$$

Using the same argument as above, we can eliminate the potential term from our equation of motion 2.2.10, resulting in

$$J^B_\nu=\frac{\xi}{\kappa}B^\mu R_{\mu\nu}. \tag{2.4.9}$$

Solving in terms of b_r (2.3.8) we get

$$\nabla^\mu B_{\mu t}-J_t=\frac{1}{r\Bigg(e^{2\gamma}\Big(b_r^2e^{-2\rho}+1\Big)\Bigg)^{3/2}}e^{2\gamma-6\rho}\Bigg(b_r^3\Bigg(-rb_r''-b_r'\Big(r\gamma'-3r\rho'+2\Big)$$
$$+b_r\Bigg(r\Big((\frac{\xi}{\kappa}-1)\gamma''+\rho''-2\rho'^2\Big)+\gamma'\Big(2(\frac{\xi}{\kappa}-1)-r(\frac{\xi}{\kappa}-2)\rho'\Big)+r\frac{\xi}{\kappa}\gamma'^2+2\rho'\Bigg)\Bigg)$$
$$+e^{2\rho}\Bigg(-rb_r'^2+b_r\Bigg(-rb_r''-b_r'\Big(r\gamma'-5r\rho'+2\Big)\Bigg)+b_r^2\Bigg(r\Big(2(\frac{\xi}{\kappa}-1)\gamma''+\rho''-3\rho'^2\Big)$$
$$+\gamma'\Big(r(3-2\frac{\xi}{\kappa})\rho'+4(\frac{\xi}{\kappa}-1)\Big)+2r\frac{\xi}{\kappa}\gamma'^2+2\rho'\Bigg)\Bigg)$$
$$+e^{4\rho}\Bigg(r(\frac{\xi}{\kappa}-1)\gamma''-(\frac{\xi}{\kappa}-1)\gamma'\Big(r\rho'-2\Big)+r\frac{\xi}{\kappa}\gamma'^2\Bigg)\Bigg) \tag{2.4.10}$$

$$\nabla^\mu B_{\mu r}-J_r=\frac{\xi}{\kappa}b_re^{-4\rho}\Bigg(-\gamma''+\gamma'\Big(\rho'-\gamma'\Big)+\frac{2\rho'}{r}\Bigg) \tag{2.4.11}$$

$$\nabla^\mu B_{\mu\theta}-J_\theta=\nabla^\mu B_{\mu\phi}-J_\phi=0. \tag{2.4.12}$$

Again repeating this process for the vector field in terms of b_t (2.3.9), we find the components

$$\nabla^\mu B_{\mu t}-J_t=\frac{1}{r}\Bigg(e^{-2(\gamma+\rho)}\Bigg(-rb_t''+b_t'\Big(r\Big(\gamma'+\rho'\Big)-2\Big)+\frac{\xi}{\kappa}b_t\Bigg(r\gamma''+\gamma'\Big(r\gamma'-r\rho'+2\Big)\Bigg)\Bigg)\Bigg) \tag{2.4.13}$$

$$\nabla^\mu B_{\mu r}-J_r=\frac{\xi}{\kappa}e^{-4\rho}\sqrt{e^{2\rho}\Big(b_t^2e^{-2\gamma}-1\Big)}\Bigg(-\gamma''+\gamma'\Big(\rho'-\gamma'\Big)+\frac{2\rho'}{r}\Bigg) \tag{2.4.14}$$

$$\nabla^\mu B_{\mu\theta}-J_\theta=\nabla^\mu B_{\mu\phi}-J_\phi=0. \tag{2.4.15}$$

2.5 Power Series Ansatz Form of the Solution

In order to find solutions from our field equations and our equations of motion, we must find the values of ρ, γ, and the vector field both in the form of b_r and b_t. To do this, we need to define an ansatz form for our solution.

All continuous functions can be written as a power series expansion around some central point. In particular, the origin can be chosen to generalize the power series to include all positive and negative terms. Such a series is called a Laurent series and is defined by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{2.5.1}$$

The advantage of this form over a more typical sum over only positive exponents is that we cannot have positive powers of r. We expect the geometry of the space to be asymptotically flat in the limit as r tends towards infinity. Any positive power of r in our sum would tend towards infinity as r goes to infinity, disrupting this local flatness; therefore, we are left only with constant and negative power terms.

$$f(z) = \sum_{n=-\infty}^{0} a_n z^n \tag{2.5.2}$$

Adapting this more general form of the series to the exact parameters given by the bumblebee field and our metric, we can use

$$f = a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \mathcal{O}(1/r^3) \tag{2.5.3}$$

as the ansatz for our solutions. We use this ansatz form for γ, ρ, and either b_r or b_t within any given calculation.

Results

3.1 Ansatz form of ρ and γ

Writing ρ and γ in the form of our ansatz, defined in Equation 2.5.3, gives

$$\rho = a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \mathcal{O}(1/r^3) \tag{3.1.1}$$

and

$$\gamma = b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \mathcal{O}(1/r^3), \tag{3.1.2}$$

where b_0 is the coefficient that allows for the rescaling of the time coordinate. We find later that this constant remains undetermined in our solutions because of this rescaling freedom.

Similarly, we can write b_r and b_t as

$$b_r = c_0 + \frac{c_1}{r} + \frac{c_2}{r^2} + \mathcal{O}(1/r^3) \tag{3.1.3}$$

and

$$b_t = c_0 + \frac{c_1}{r} + \frac{c_2}{r^2} + \mathcal{O}(1/r^3). \tag{3.1.4}$$

It is important to note here that b_r and b_t both use c as their coefficient because the results of the vector field in terms of each were solved independently of the other. The c_0 coefficients determine the asymptotic behavior of the vector field; in order to have a uniform field asymptotically, they would need to have constant values.

3.2 Solving for Values of ρ and γ

The form of the Einstein equation that we used in 2.4.2 requires all the components of $G_{\mu\nu}$ to equal zero, meaning each term in our series must also equal zero. Therefore, we expect to find values for the coefficients such that the numerators of each term become zero.

To solve for our coefficients, we began by attempting to simplify our series equations by looking for coefficients whose values we could substitute into the code without the risk of losing any solutions. In effect, we wanted to find coefficients that have exactly one value, without which at least one of our terms would not go to zero. The full form of our series equations and the exact process by which they were simplified is presented in appendices A.2 and A.3.

3.2.1 Solutions for Vector Field Written in Terms of b_r

In the case of the series defined in terms of b_r, γ, and ρ, we defined $c_0 = 0$ so as to have a timelike solution where b_r goes to 0 asymptotically. We were able to determine that $a_0 = 0$ must be true, as does $a_1 = -b_1$ (detailed process in appendix A.2).

Implementing these simplifications, we were able to use our Mathematica code to solve for the other coefficients. This yielded the results,

$$a_2 = 0 \qquad\qquad b_1 = 0 \qquad\qquad b_2 = 0 \qquad\qquad c_1 = 0 \tag{3.2.1}$$

b_0 remains undetermined which represents that we are able to rescale our time coordinate, as we would expect. All instances of c_2 were eliminated by the substitution of $a_1 = -b_1$, meaning it, too, remains undetermined.

3.2.2 Solutions for Vector Field Written in Terms of b_t

The same process was used to solve for the vector field in terms of b_t (appendix A.3), where the substitutions used to simplify the equations before they were solved using Mathematica were $a_0 = 0, a_1 = -b_1$, and $c_0 = e^{b_0}$. In this case, the code yielded

$$\begin{aligned}
a_2 &= \frac{e^{-2b_0}\left(8e^{2b_0}b_1^2\frac{\xi}{\kappa}^2 - 12e^{2b_0}b_1^2\frac{\xi}{\kappa} + 8e^{2b_0}b_1^2 + 2c_1^2\frac{\xi}{\kappa}^2 + 3c_1^2\frac{\xi}{\kappa} - 2c_1^2\right)}{4\left(2\frac{\xi}{\kappa}^2 - 3\frac{\xi}{\kappa} + 2\right)} \\
b_2 &= -\frac{e^{-2b_0}\left(4e^{2b_0}b_1^2\frac{\xi}{\kappa}^2 - 6e^{2b_0}b_1^2\frac{\xi}{\kappa} + 4e^{2b_0}b_1^2 + 2c_1^2\frac{\xi}{\kappa} - c_1^2\right)}{2\left(2\frac{\xi}{\kappa}^2 - 3\frac{\xi}{\kappa} + 2\right)} \\
c_2 &= -\frac{e^{-b_0}c_1^2\frac{\xi}{\kappa}(2\frac{\xi}{\kappa} - 1)}{2\left(2\frac{\xi}{\kappa}^2 - 3\frac{\xi}{\kappa} + 2\right)}.
\end{aligned} \tag{3.2.2}$$

Discussion

The solutions presented in 3.2.1, and 3.2.2 have different values for the c_0 coefficient. As mentioned in Section 3.1, the values for c_0 describe the asymptotic behavior of the different components of the vector field. Therefore, it is possible for them different to have different values. However, we would expect all other coefficients' values to be the same. One possible solution to the results shown in 3.2.2 is every coefficient equalling zero, like those in 3.2.1, which would match our expectations. However, the square root in Equation 2.3.9 implies the potential existence of fractional powers of r, which are not included in the current form of our ansatz.

While further work is needed to confirm our results, especially those presented in 3.2.2, our results for 2.3.8, as seen in 3.2.1, suggest that all defined coefficients of the series to the third order other than c_0 are zero. It is possible that higher-order terms have nonzero coefficients. In order to verify our result, we attempted to solve the field equations and equations of motion directly. To demonstrate these calculations, we will be using the vector field in terms of b_r.

We begin by assuming that $b_r \neq 0, \gamma \neq 0$, and $\rho \neq 0$, in effect, preserve that the bumblebee vector field does exist and that our equations do have solutions other than those that were already found. We then can consider the $\nabla^\mu B_{\mu r} - J_r$ 2.4.11 component as it is the simplest

$$\nabla^\mu B_{\mu r} - J_r = \frac{\xi}{\kappa} b_r e^{-4\rho}\bigg(-\gamma'' + \gamma'\Big(\rho' - \gamma'\Big) + \frac{2\rho'}{r}\bigg) = 0. \tag{4.0.3}$$

There are several ways for this statement to be true; however, the one that does not violate our assumptions is when

$$\gamma'' = \gamma'\Big(\rho' - \gamma'\Big) + \frac{2\rho'}{r}. \tag{4.0.4}$$

As all our field equations and equations of motion must equal zero, we can now implement this form of γ'' as a simplification in our other equations. From there, we can use our program to solve other equations looking for exclusively real solutions. Given this simplification, our code could not find any real solutions, suggesting that one or more of our assumptions cannot hold. Proving these solutions analytically would take substantial effort and is beyond the scope of this work. However, our code being unable to find any real solutions without violating our initial assumptions does support the conclusions found earlier.

The previous work in this section suggests that $\gamma = \rho = 0$ is likely the only solution to our Einstein equations. If that is the case, then we can substitute them back into our metric 2.3.1 to get

$$g_{\mu\nu} = \mathrm{diag}(-e^{2(0)}, e^{2(0)}, r^2, r^2 \sin^2\theta), \tag{4.0.5}$$

which simplifies to

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2 \sin^2\theta). \tag{4.0.6}$$

This is the metric for flat space written in spherical coordinates.

If the only solutions to our Einstein equations are those associated with flat spacetime, this presents a problem for the model we are considering, as it suggests that there are no black hole solutions for an asymptotically timelike bumblebee field. Our current understanding of general relativity dictates that the existence of a black hole necessitates a curvature of spacetime. Our solutions predict a spacetime that is perfectly flat everywhere, which by definition cannot contain a black hole. Given that such an empty space is incompatible with the universe as we know it to exist, our model is likely not viable.

Conclusion

The results of our work solving the Einstein equations in the presence of a bumblebee vector field cast doubt on the viability of this model. Our solutions suggest that this model can only exist in a perfectly flat spacetime. Such a situation is fundamentally at odds with our observations of the universe. Therefore, if these results hold, the bumblebee model is not a physically viable model.

There are some significant limits to the work done for this project, and future work must be conducted to test this model further and validate our results. Most importantly, our ansatz will need to be redefined to account for possible fractional powers. Along with this, exact analytical or numerical solutions should be found for the Einstein equations to prove that the only solutions are the ones shown in this work. Finally, a further study should be conducted to solve the Einstein equation with the inclusion of the matter sector. It is also possible that solutions could be found using this model if we relaxed some of our constraints, including looking for possible solutions that might exist outside the potential minimum or those that do not require strict spherical symmetry.

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Appendices

A.1 Overview of Method

We created a program using Mathematica and the xTensor package which allows for solving the Einstein tensor and the equations of motion to find solutions for ρ and γ in terms of their ansatz equations.

We begin by defining a four-dimensional manifold, reserving the letters "a" through "p" to use as indices. We also define ρ and γ to be scalar functions. We then define a Birkhoff metric within this manifold as a diagonal metric with components shown in equation 2.3.1.

To define the background field, we define b_r and b_t as scalar functions and $\pm b^2$ as a constant symbol. We then express the background vector field in terms of these symbols. It is important to note here that we create two code files that run identical processes, with one defining the background field entirely in terms of b_r, and the other entirely in terms of b_t, as made possible by 2.3.8.

We then calculate the covariant derivative of the metric and the Ricci tensor. Using these and equation 2.2.7, we calculate the stress-energy tensor using a constant symbol for $\frac{\xi}{\kappa}$. We use equation 2.2.8 to find the Einstein tensor. We then calculate the equations of motion using equation 2.2.10. We can simplify the resulting equations by setting $\pm b^2 = 1$ and setting our constants to unity.

A.2 Ansatz forms for γ, ρ, and b_r

In this section, we provide the ansatz forms of γ, ρ, and b_r (ref eqs here) and detail the method by which they were simplified. To do this, we rewrite Equations 2.4.3, 2.4.4, 2.4.5, 2.4.10, and 2.4.11 using the series forms of γ, ρ, and b_r which result in

$$\begin{aligned} G_{tt} - \kappa T_{tt} = {} & \frac{e^{-2(a_0+b_0)}\left(e^{2a_0}-1\right)}{r^2} - \frac{\left(e^{2a_0}-1\right)\left(2b_1 e^{-2(a_0+b_0)}\right)}{r^3} + \\ & \frac{e^{-2(2a_0+b_0)}\left(e^{2a_0}\left(b_1^2\left(4e^{2a_0}-2\frac{\xi}{\kappa}-5\right)-4b_2\left(e^{2a_0}+\frac{\xi}{\kappa}-1\right)+4a_1^2+2a_1b_1\frac{\xi}{\kappa}-4a_2\right)+2c_1^2\frac{\xi}{\kappa}\right)}{2r^4} \\ & + \mathcal{O}\left(\frac{1}{r^5}\right) \end{aligned} \tag{A.2.1}$$

$$G_{rr} - \kappa T_{rr} = \frac{e^{-4a_0} - e^{-2a_0}}{r^2} + \frac{2e^{-4a_0}\Bigg(-b_1 + a_1\Big(e^{2a_0} - 2\Big)\Bigg)}{r^3} + \frac{e^{-6a_0}\Bigg(e^{2a_0}\Big(16a_1^2 + 16a_1 b_1 - 8a_2 + 8b_2 + b_1^2\Big) - 4e^{4a_0}\Big(a_1^2 - a_2\Big) - 2c_1^2\frac{\xi}{\kappa}\Bigg)}{2r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.2.2}$$

$$G_{\theta\theta} - \kappa T_{\theta\theta} = \frac{\frac{1}{2}e^{-2a_0}\Bigg(-4a_1^2 - 6a_1 b_1 + 4a_2 + b_1^2 + 8b_2\Bigg) + c_1^2\frac{\xi}{\kappa}e^{-4a_0}}{r^6} + \frac{e^{-2a_0}(a_1 + b_1)}{r^5} + \mathcal{O}\Bigg(\frac{1}{r^7}\Bigg) \tag{A.2.3}$$

$$\nabla^\mu B_{\mu t} - J_t = \frac{e^{-4a_0}\Bigg(e^{2a_0}\Bigg(a_1(b_1 - b_1\frac{\xi}{\kappa}) + b_1^2\frac{\xi}{\kappa} + (2b_2)(\frac{\xi}{\kappa} - 1)\Bigg) - c_1^2\Bigg)}{\sqrt{e^{2b_0}}r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.2.4}$$

$$\nabla^\mu B_{\mu r} - J_r = -\frac{2c_1\frac{\xi}{\kappa}e^{-4a_0}(a_1 + b_1)}{r^4} + \frac{\frac{\xi}{\kappa}e^{-4a_0}\Bigg(c_1(8a_1^2 + 9a_1 b_1 - 4a_2 - b_1^2 - 6b_2) - 2c_2(a_1 + b_1)\Bigg)}{r^5} + \mathcal{O}\Bigg(\frac{1}{r^6}\Bigg) \tag{A.2.5}$$

Given the form of the Einstein equation in 2.4.2 that we used requires all the components of $G_{\mu\nu}$ to equal zero we expect to find values for the coefficients such that every term of these series also go to zero. To that end, we look for numerators that require a set value for one of our coefficients to become zero.

In the case of the above equations, the $\frac{1}{r^2}$ term of $G_{rr} - \kappa T_{rr}$ is

$$\frac{e^{-4a_0} - e^{-2a_0}}{r^2}.$$

we can see from this term that $a_0 = 0$ is the only possible value that allows the whole term to be zero. We can then implement this value as a simplification to our equations resulting in the series forms of γ, ρ, and b_r simplifying to

$$G_{tt} - \kappa T_{tt} = \frac{e^{-2b_0}\Bigg(4a_1^2 + 2a_1 b_1\frac{\xi}{\kappa} - 4a_2 - 2b_1^2\frac{\xi}{\kappa} - b_1^2 - 4b_2\frac{\xi}{\kappa} + 2c_1^2\frac{\xi}{\kappa}\Bigg)}{2r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.2.6}$$

$$G_{rr} - \kappa T_{rr} = -\frac{2(a_1 + b_1)}{r^3} + \frac{6a_1^2 + 8a_1 b_1 - 2a_2 + \frac{b_1^2}{2} - 4b_2 - c_1^2\frac{\xi}{\kappa}}{r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.2.7}$$

$$G_{\theta\theta} - \kappa T_{\theta\theta} = \frac{a_1 + b_1}{r^5} + \frac{-2a_1^2 - 3a_1 b_1 + 2a_2 + \frac{b_1^2}{2} + 4b_2 + c_1^2\frac{\xi}{\kappa}}{r^6} + \mathcal{O}\Bigg(\frac{1}{r^7}\Bigg) \tag{A.2.8}$$

$$\nabla^\mu B_{\mu t} - J_t = \frac{a_1(b_1 - b_1\frac{\xi}{\kappa}) + b_1^2\frac{\xi}{\kappa} + 2b_2(\frac{\xi}{\kappa} - 1) - c_1^2}{\sqrt{e^{2b_0}}r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.2.9}$$

$$\nabla^\mu B_{\mu r} - J_r = -\frac{2c_1\frac{\xi}{\kappa}(a_1+b_1)}{r^4} + \frac{\frac{\xi}{\kappa}\Big(8a_1^2c_1 + 9a_1b_1c_1 - 2a_1c_2 - 4a_2c_1 - b_1^2c_1 - 2b_1c_2 - 6b_2c_1\Big)}{r^5} + \mathcal{O}\Big(\frac{1}{r^6}\Big). \tag{A.2.10}$$

In these simplified forms of the equations, we find that the $\frac{1}{r^3}$ term of $G_{rr} - \kappa T_{rr}$ and the $\frac{1}{r^5}$ term of $G_{\theta\theta} - \kappa T_{\theta\theta}$ have an $(a_1 + b_1)$ as the primary component of their numerator, implying that $a_1 = -b_1$. Using this, we find that

$$G_{tt} - \kappa T_{tt} = \frac{e^{-2b_0}\Big(-4a_2 + b_1^2(3 - 4\frac{\xi}{\kappa}) + 2\frac{\xi}{\kappa}\Big(c_1^2 - 2b_2\Big)\Big)}{2r^4} + \mathcal{O}\Big(\frac{1}{r^5}\Big) \tag{A.2.11}$$

$$G_{rr} - \kappa T_{rr} = \frac{-2a_2 - \frac{3b_1^2}{2} - 4b_2 - c_1^2\frac{\xi}{\kappa}}{r^4} + \mathcal{O}\Big(\frac{1}{r^5}\Big) \tag{A.2.12}$$

$$G_{\theta\theta} - \kappa T_{\theta\theta} = \frac{2a_2 + \frac{3b_1^2}{2} + 4b_2 + c_1^2\frac{\xi}{\kappa}}{r^6} + \mathcal{O}\Big(\frac{1}{r^7}\Big) \tag{A.2.13}$$

$$\nabla^\mu B_{\mu t} - J_t = \frac{b_1^2(2\frac{\xi}{\kappa} - 1) + 2b_2(\frac{\xi}{\kappa} - 1) - c_1^2}{\sqrt{e^{2b_0}}r^4} + \mathcal{O}\Big(\frac{1}{r^5}\Big) \tag{A.2.14}$$

$$\nabla^\mu B_{\mu r} - J_r = \frac{-2c_1\frac{\xi}{\kappa}(2a_2 + b_1^2 + 3b_2)}{r^5} + \mathcal{O}\Big(\frac{1}{r^6}\Big) \tag{A.2.15}$$

This is the final form the series equations were reduced to before being solved with Mathematica.

A.3 Ansatz forms for γ, ρ, and b_t

In this section, we provide the ansatz forms of γ, ρ, and b_t (3.1.2, 3.1.1 3.1.4) and detail the method by which they were simplified. To do this, we rewrite Equations 2.4.6, 2.4.7, 2.4.8, 2.4.13, and 2.4.14 using the series forms of γ, ρ, and b_t which result in

$$\begin{aligned} G_{tt} - \kappa T_{tt} = {} & \frac{e^{-4b_0}\frac{\xi}{\kappa}\Big(e^{2b_0} - c_0^2\Big)}{r^2} - \frac{2\Big(e^{-4b_0}b_1\frac{\xi}{\kappa}\Big(e^{2b_0} - c_0^2\Big)\Big)}{r^3} \\ & + \frac{1}{2r^4}\Bigg(e^{-4b_0}\Bigg(-4e^{2b_0}\Bigg(a_1^2(\frac{\xi}{\kappa} - 1) - \frac{\xi}{\kappa}\Big(a_2 + b_1^2 - b_2\Big) + a_2\Bigg) \\ & + 2c_0\frac{\xi}{\kappa}\Big(c_0\Big(2a_1^2 + 5a_1b_1 - 2(a_2+b_2) - b_1^2\Big) + 2c_2\Big) - 8c_0c_1\frac{\xi}{\kappa}(a_1+b_1) + c_1^2(2\frac{\xi}{\kappa} - 1)\Bigg)\Bigg) \\ & + \mathcal{O}\Big(\frac{1}{r^5}\Big) \end{aligned} \tag{A.3.1}$$

$$
\begin{aligned}
G_{rr} - \kappa T_{rr} = {} & \frac{\frac{\xi}{\kappa}\left(e^{-2b_0}c_0^2 - 1\right)}{r^2} + \frac{-2e^{-2b_0}c_0^2\frac{\xi}{\kappa}(3a_1 + 2b_1) + 6a_1\frac{\xi}{\kappa} - 2a_1 + 4b_1\frac{\xi}{\kappa} - 2b_1}{r^3} \\
& + \left(e^{-2b_0}\left(-2e^{2b_0}\left(2a_1^2(8\frac{\xi}{\kappa} - 3) + a_1b_1(17\frac{\xi}{\kappa} - 8) - \frac{\xi}{\kappa}\left(8a_2 + b_1^2 + 10b_2\right) + 2a_2 + 4b_2\right)\right.\right. \\
& \left.\left. + 2c_0\frac{\xi}{\kappa}(c_0(a_1 + b_1)(16a_1 + 5b_1) - 8c_0(a_2 + b_2) - 2c_2) - 8c_0c_1\frac{\xi}{\kappa}(a_1 + b_1) + c_1^2(1 - 2\frac{\xi}{\kappa})\right)\right)\frac{1}{2r^4} \\
& + \mathcal{O}\left(\frac{1}{r^5}\right)
\end{aligned}
\tag{A.3.2}
$$

$$
\begin{aligned}
G_{\theta\theta} - \kappa T_{\theta\theta} = {} & \frac{(a_1 + b_1)\left(\frac{\xi}{\kappa}\left(e^{-2b_0}c_0^2 - 1\right) + 1\right)}{r^5} \\
& + \frac{1}{2r^6}\left(e^{-2b_0}\left(2e^{2b_0}(\frac{\xi}{\kappa} - 1)\left(2a_1^2 + 3a_1b_1 - 2a_2 - b_1^2 - 4b_2\right)\right.\right. \qquad \text{(A.3.3)} \\
& \left.\left. + 4c_0\frac{\xi}{\kappa}\left(c_0\left(-(a_1 + b_1)^2 + a_2 + b_2\right) + c_2\right) + 2c_0c_1\frac{\xi}{\kappa}(a_1 + b_1) + c_1^2(2\frac{\xi}{\kappa} - 1)\right)\right) \\
& + \mathcal{O}\left(\frac{1}{r^7}\right)
\end{aligned}
\tag{A.3.4}
$$

$$
\nabla^\mu B_{\mu t} - J_t = \frac{e^{-2b_0}\left(c_0\frac{\xi}{\kappa}\left(-a_1b_1 + b_1^2 + 2b_2\right) + c_1(a_1 + b_1) - 2c_2\right)}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right)
\tag{A.3.5}
$$

$$
\begin{aligned}
\nabla^\mu B_{\mu r} - J_r = {} & -\frac{2\left(\frac{\xi}{\kappa}(a_1 + b_1)\sqrt{e^{-2b_0}c_0^2 - 1}\right)}{r^3} \\
& + \frac{1}{r^4\left(e^{2b_0} - c_0^2\right)}\left(\frac{\xi}{\kappa}\sqrt{e^{-2b_0}c_0^2 - 1}\left(e^{2b_0}\left(6a_1^2 + 7a_1b_1 - 4a_2 - b_1^2 - 6b_2\right)\right.\right. \\
& \left.\left. + c_0\left(2c_1(a_1 + b_1) - c_0\left(6a_1^2 + 9a_1b_1 - 4a_2 + b_1^2 - 6b_2\right)\right)\right)\right) \\
& + \mathcal{O}\left(\frac{1}{r^5}\right).
\end{aligned}
\tag{A.3.6}
$$

Given the form of the Einstein equation in 2.4.2 that we used requires all the components of $G_{\mu\nu}$ to equal zero we expect to find values for the coefficients such that every term of these series also go to zero. To that end, we look for numerators that require a set value for one of our coefficients to become zero.

In the case of the above equations, the $\frac{1}{r^2}$ term of $G_{tt} - \kappa T_{tt}$ is

$$
\frac{e^{-4b_0}\frac{\xi}{\kappa}\left(e^{2b_0} - c_0^2\right)}{r^2}.
\tag{A.3.7}
$$

In order for the numerator of this term to equal zero $e^{2b_0} - c_0^2$ must equal zero. This implies that $c_0 = e^{b_0}$.We can then implement this value as a simplification to our equations resulting in the series forms of γ, ρ, and b_t simplifying to

$$G_{tt} - \kappa T_{tt} = \frac{1}{2r^4}\Bigg(e^{-4b_0}\Bigg(2e^{b_0}\Bigg(e^{b_0}\Bigg(2a_1^2 + \frac{\xi}{\kappa}\Bigg(5a_1b_1 + b_1^2 - 4b_2\Bigg) - 2a_2\Bigg) + 2c_2\frac{\xi}{\kappa}\Bigg) - 8e^{b_0}c_1\frac{\xi}{\kappa}(a_1+b_1) + c_1^2(2\frac{\xi}{\kappa} - 1)\Bigg)\Bigg) + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.3.8}$$

$$G_{rr} - \kappa T_{rr} = -\frac{2(a_1+b_1)}{r^3} + \frac{1}{r^4}\Bigg(6a_1^2 - 2e^{-b_0}\frac{\xi}{\kappa}(2c_1(a_1+b_1)+c_2) + 4a_1b_1(\frac{\xi}{\kappa}+2) - 2a_2 + \frac{1}{2}e^{-2b_0}c_1^2(1-2\frac{\xi}{\kappa}) + 6b_1^2\frac{\xi}{\kappa} + 2b_2(\frac{\xi}{\kappa} - 2)\Bigg) + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.3.9}$$

$$G_{\theta\theta} - \kappa T_{\theta\theta} = \frac{a_1+b_1}{r^5} + \frac{1}{r^6}\Bigg(-2a_1^2 + e^{-b_0}\frac{\xi}{\kappa}(c_1(a_1+b_1)+2c_2) - a_1b_1(\frac{\xi}{\kappa}+3) + 2a_2 + \frac{1}{2}e^{-2b_0}c_1^2(2\frac{\xi}{\kappa} - 1) + b_1^2(1-3\frac{\xi}{\kappa}) - 2b_2(\frac{\xi}{\kappa} - 2)\Bigg) + \mathcal{O}\Bigg(\frac{1}{r^7}\Bigg) \tag{A.3.10}$$

$$\nabla^\mu B_{\mu t} - J_t = \frac{e^{-2b_0}\Bigg(-e^{b_0}\frac{\xi}{\kappa}(b_1(a_1-b_1)-2b_2) + c_1(a_1+b_1) - 2c_2\Bigg)}{r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.3.11}$$

$$\nabla^\mu B_{\mu r} - J_r = -2\Bigg(\frac{1}{r}\Bigg)^{7/2}\Bigg(\sqrt{2}\sqrt{r}\frac{\xi}{\kappa}(a_1+b_1)\sqrt{\frac{e^{-b_0}\Big(c_1 - e^{b_0}b_1\Big)}{r}}\Bigg) + \mathcal{O}\Bigg(\frac{1}{r^{9/2}}\Bigg). \tag{A.3.12}$$

In these simplified forms of the equations, we find that the $\frac{1}{r^3}$ term of $G_{rr} - \kappa T_{rr}$ and the $\frac{1}{r^5}$ term of $G_{\theta\theta} - \kappa T_{\theta\theta}$ have an $(a_1 + b_1)$ as the primary component of their numerator, implying that $a_1 = -b_1$. Using this, we find that

$$G_{tt} - \kappa T_{tt} = \frac{e^{-4b_0}\Bigg(4e^{b_0}\Bigg(e^{b_0}\Bigg(-a_2 - 2\frac{\xi}{\kappa}\Bigg(b_1^2+b_2\Bigg) + b_1^2\Bigg) + c_2\frac{\xi}{\kappa}\Bigg) + c_1^2(2\frac{\xi}{\kappa} - 1)\Bigg)}{2r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.3.13}$$

$$G_{rr} - \kappa T_{rr} = \frac{-2a_2 + \frac{1}{2}e^{-2b_0}c_1^2(1-2\frac{\xi}{\kappa}) - 2e^{-b_0}c_2\frac{\xi}{\kappa} + 2b_1^2(\frac{\xi}{\kappa} - 1) + 2b_2(\frac{\xi}{\kappa} - 2)}{r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.3.14}$$

$$G_{\theta\theta} - \kappa T_{\theta\theta} = \frac{e^{-2b_0}\Bigg(4e^{b_0}\Bigg(e^{b_0}\Bigg(a_2 - \frac{\xi}{\kappa}\Bigg(b_1^2+b_2\Bigg) + b_1^2 + 2b_2\Bigg) + c_2\frac{\xi}{\kappa}\Bigg) + c_1^2(2\frac{\xi}{\kappa} - 1)\Bigg)}{2r^6} + \mathcal{O}\Bigg(\frac{1}{r^7}\Bigg) \tag{A.3.15}$$

$$\nabla^\mu B_{\mu t} - J_t = \frac{e^{-2b_0}\Bigg(2e^{b_0}\frac{\xi}{\kappa}\Bigg(b_1^2+b_2\Bigg) - 2c_2\Bigg)}{r^4} + \mathcal{O}\Bigg(\frac{1}{r^5}\Bigg) \tag{A.3.16}$$

$$\nabla^\mu B_{\mu r} - J_r = \mathcal{O}\Bigg(\frac{1}{r^{9/2}}\Bigg). \tag{A.3.17}$$

This is the final form the series equations were reduced to before being solved with Mathematica.